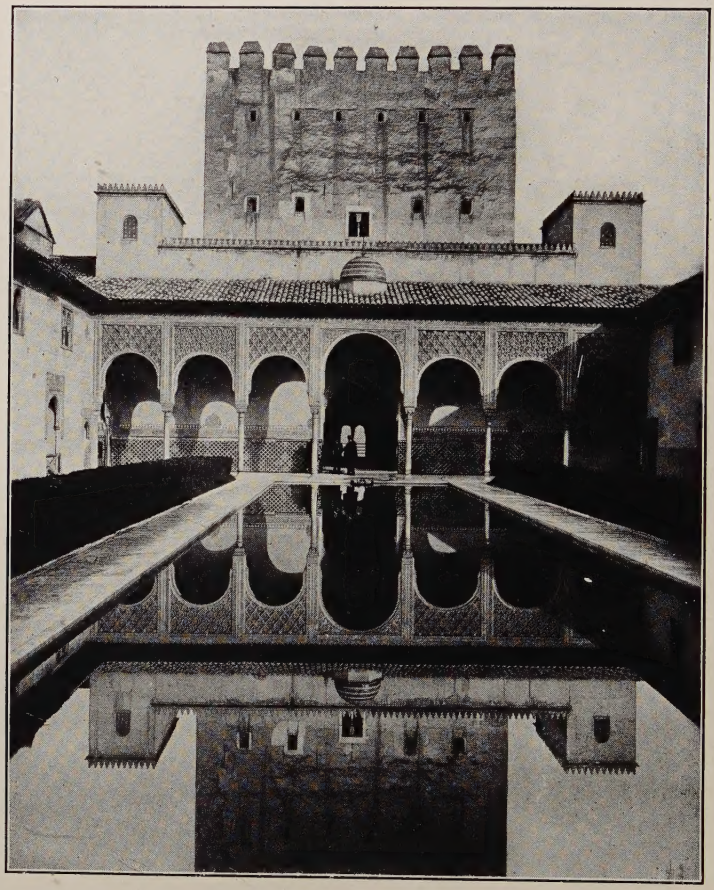
NEW PLANE GEOMETRY

DURELL AND ARNOLD

WITH
REVIEW AND DIAGNOSTIC
TESTS



THE COURT OF MYRTLES, THE ALHAMBRA

The Moors in Spain produced the lace-like filigree in their stone palaces through the repetition of simple geometric patterns. The beauty of symmetrical figures was achieved also by means of reflections in quiet pools. (*Publishers Photo Service*)

Henry J. Schillmoeller

NEW PLANE GEOMETRY

BY

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WITH
REVIEW AND DIAGNOSTIC
TESTS

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PREFACE

THIS revision of the Durell and Arnold PLANE GEOMETRY has been based mainly on the following ideas:

1. While certain features of modern life, such as the kodak, moving picture, and automobile, have a strong appeal for the average high school pupil, in competition with his studies, these same conditions make the pupil more alert and eager to learn, especially if a subject is made simple and easy to appreciate at the outset.
2. Real success in the study of geometry is in proportion to the ability acquired to solve original exercises.

The recent revisions of leading syllabi in plane geometry are based on these principles. Regarding the solution of originals one of them says:

"Good training in plane geometry should enable pupils to solve rapidly at sight the simpler kinds of original exercises. . . . It should be remembered always that pupils acquire facility in solution more readily by solving a large number of very simple exercises than by attempting a small number of exercises of a difficult character."

The recent College Entrance Examination Board syllabus says essentially the same and adds that in the future, in the College Board papers on geometry, one third of the questions will be "book propositions," another third will be "easy originals," and the remaining third will consist of "originals, which, though still not of excessive difficulty, are designed to test the powers of the better candidates."

In accordance with this view, more than 150 new, simple, and varied originals have been added to those already in the Durell and Arnold textbook, and some of the more difficult and technical originals have been omitted. Almost half of the new exercises, most of them adapted to sight work, have been inserted in Book One.

Advantage has been taken of the opportunity afforded by the present revision to improve some details of the book in the light of the experience of many teachers who have been using the text under a variety of circumstances. These changes include a restatement of some of the definitions and a more detailed presentation of parts of some of the proofs.

While making the changes outlined above, the authors have retained those distinctive features which have made the book a success in the past, as:

1. The simple and informal introduction to geometry, with easy exercises, including drawing problems, to develop a real grasp of the subject by the pupil.
2. The careful selection and presentation of the first nine propositions of Book One so that these first propositions shall not be a stumbling block to the beginner. No indirect demonstration is called for until the pupil has become familiar with direct proofs and has worked many simple originals.
3. The arrangement in parallel columns of the steps and reasons in the proofs.
4. The requiring of the pupil to supply an increasing number of reasons in the proofs as the book progresses.
5. The group method of teaching originals.
6. The free use of algebraic and other methods of analysis.
7. The simple and valuable exercises in practical applications of geometry and in the construction of ornamental designs.

All the propositions listed in recent important syllabi are included in the book. In the appendix a brief treatment of the method of limits is presented, with the application of this method to a representative proposition.

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SYMBOLS AND ABBREVIATIONS

$+$ *plus* or *increased by.*
$-$ *minus* or *diminished by.*
$\times$ *multiplied by.*
$\div$ *divided by.*
$=$ *equals; is* (or *are*) *equal to.*
$\cong$ *is congruent to.*
$\overset{m}{=}$ *is measured by.*
$>$ *is* (or *are*) *greater than.*
$<$ *is* (or *are*) *less than.*
$\sim$ *is* (or *are*) *similar to.*
$\therefore$ *therefore.*
$\perp$ *perpendicular, perpendicular to,* or *is perpendicular to.*
$\perp\!\!\!\perp$ *perpendiculars.*
$\parallel$ *parallel* or *is parallel to.*
$\angle$, $\measuredangle$ *angle, angles.*
$\triangle$, $\blacktriangle$ *triangle, triangles.*
$\square$, $\boxed{S}$ *parallelogram, parallelograms.*
$\odot$, $\textcircled{S}$ *circle, circles.*
$\frown$ (as in $\overset{\frown}{AB}$) *arc.*

Adj., *adjacent.*
Alt., *alternate.*
Ax., *axiom.*
Comp., *complement.*
Constr., *construction.*
Cor., *corollary.*
Corr., *corresponding.*
Def., *definition.*
Ex., *exercise* or *example.*
Ext., *exterior.*
Fig., *figure.*
Geom., *geometry.*
Hyp., *hypothesis.*
Ident., *identity.*
Ineq., *inequality.*
Int., *interior.*
Opp., *opposite.*
Post., *postulate.*
Prop., *proposition.*
Rt., *right.*
St., *straight.*
Sug., *suggestion.*
Sup., *supplement.*

Q.E.D. *quod erat demonstrandum;* that is, which was to be proved.
Q.E.F. *quod erat faciendum;* that is, which was to be made.

A few other abbreviations and symbols will be introduced and explained later.

NEW PLANE GEOMETRY

DEFINITIONS AND FIRST PRINCIPLES

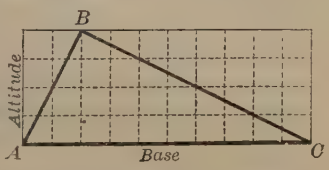
1. Some geometry which you already know.—From your study of arithmetic, and in other ways, you have already learned some of the most important properties of the straight line, circle, triangle, cube, sphere, and a few other similar objects.

You know that a solid has three dimensions, viz.: length, breadth, and thickness. How many dimensions has a surface? A line? A point?

2. Efficient methods of treating geometric objects.—In dealing with objects like those just named in § 1, you have learned that often it is an advantage to use certain methods rather than others.

EXERCISES: GROUP 1

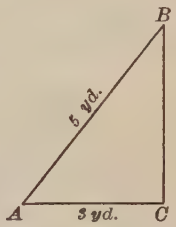
Ex. 1. Obtain the area of the triangle ABC by counting the small squares in the triangle (piecing together parts of squares). Also obtain the same area by multiplying the number of linear spaces in the base by the number in the altitude, and taking one half the product. Estimate how many times as much work you did in the first process as in the second.



Ex. 2. The number of bushels of wheat in a given bin might be determined by filling a bushel measure with wheat from the bin, time after time, till the wheat is exhausted, and counting the number of times the bushel measure is used. The number of bushels might be determined also by measuring (in feet) the three dimensions of the

bin and dividing their product by the number of cubic feet in a bushel ($1\frac{1}{4}$ cu. ft. approx. = 1 bu.). Compare the amount of work in the two processes, assuming that the bin is a large one.

Ex. 3. A given ladder AB is 5 yd. long, and its foot A is 3 yd. distant from the wall BC. In order to determine BC, which is easier: to measure BC, or to use the following computation?



No. yd. in $BC = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = 4$.

For the following exercises the student should have a ruler with an edge divided into inches and eighths of an inch.

Ex. 4. By folding a piece of paper, form a straight edge. With the aid of this straight edge, draw four lines of unequal length. Estimate the length of each of these lines and then measure the length of each to the nearest $\frac{1}{8}$ inch. Tabulate your results as follows:

Line	Estimated Length	Measured Length	Error
1			
2			
3			
4			

Ex. 5. With the paper straight edge, draw a line which you estimate to be twice as long as line 1. Check your estimate by measuring with the ruler the length of the line drawn.

Ex. 6. With the paper straight edge, draw a line and by estimate mark the middle point of the line. Test your estimate by use of your ruler.

Ex. 7. With the straight edge, draw a line which you estimate to be equal to the sum of lines 1 and 2. Check your estimate by use of the ruler.

Ex. 8. With the straight edge, draw a line which you estimate to be equal to the difference between lines 2 and 3. Check by use of the ruler.

Ex. 9. With the straight edge, draw a line and by estimate mark points which divide the line into three equal parts. Check your estimate with the ruler.

Ex. 10. In each of the following figures, estimate whether the line a is longer or shorter than the line b. Check your estimate with the ruler.

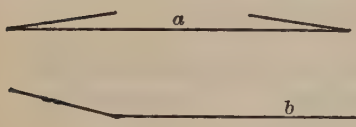


FIG. 1

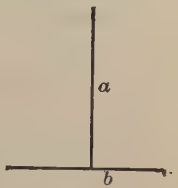


FIG. 2

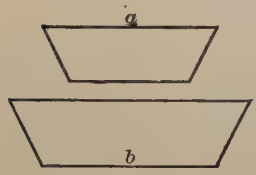


FIG. 3

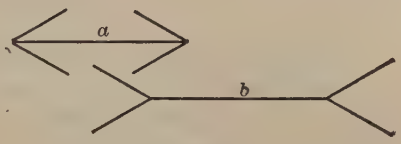


FIG. 4

Ex. 11. Of the geometric objects—the solid, surface, line, and point—which is the boundary of a solid? Of a surface? Of a line?

3. Geometry is the study of the most efficient methods of dealing with the shape, size, and position of objects.

4. The **fundamental geometric objects** are the point, line, surface, and solid. For the present, you understand sufficiently well what these are without studying formal definitions of them.

5. A **geometric figure** is a point, line, surface, or solid, or any combination of these.

Thus in arithmetic you have already used geometric figures like the following:

6. The **form** or **shape** of a figure is determined by the relative position of the various parts of the figure.

7. Similar geometric figures are those which have the same *shape.*

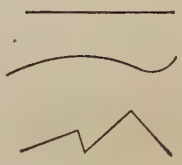
Equivalent figures are those which have the same *size.*

Congruent figures are those which have the same *shape* and *size,* and can, therefore, be made to coincide.

"Congruent figures" are often termed "equal figures."

8. A **point** is represented to the eye by a **dot** and is named by a letter affixed to the dot; as . A, called the point A.

LINES



9. Straight line.—You already know that a straight line is the shortest line that can be drawn joining two points, and you are familiar with some of the other useful properties of a straight line.

The word "line" is often used instead of "straight line."

10. A **curved line** is a line no portion of which is straight.

The word "curve" is often used for "curved line."

11. A **broken line** is a line made up of different straight lines.

EXERCISES: GROUP 2

Ex. 1. On a sheet of paper or on the blackboard, locate a point and draw two lines through this point.

Ex. 2. Can the lines drawn in Ex. 1 meet at another point?

Ex. 3. Can more than two lines be drawn through the given point? How many?

Ex. 4. On a flat surface can you draw two lines which would not meet, however far the lines are produced?

Ex. 5. On a flat surface, if you can, draw three lines, no two of which will meet no matter how far they are produced.

Ex. 6. On a flat surface, how many lines can be drawn so that they will not meet?

Ex. 7. If, on the given surface, a line is drawn meeting one of the lines in Ex. 6, how many of the other lines will it meet if it is extended?

Ex. 8. On a sheet of paper or on the blackboard, locate two points and draw a line passing through both of these points.

Ex. 9. Can another line be drawn passing through the two points of Ex. 8?

Ex. 10. Can more than one curved line be drawn passing through these two points?

Ex. 11. How many points are required to fix the position of a line?

Ex. 12. On a piece of paper or on the blackboard, draw a line along the edge of your ruler; then turn the ruler over and fit the same edge to the line. Will this test the edge of the ruler for straightness?

Ex. 13. Fold a sheet of paper. Test the edge of the fold in the same way that you tested the edge of the ruler in Ex. 12.

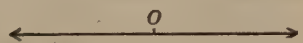
Ex. 14. Fit the edge of your ruler to the edge made by the fold of the paper (Ex. 13). How many points in the two edges must be made to coincide, in order that the two edges shall coincide throughout?

Ex. 15. Why are two sights necessary for a gun?

Ex. 16. In order to hold a straight iron rod in a given position, at how many points is it necessary to fasten the rod rigidly?

Ex. 17. When a farmer wishes to set out three or more apple trees in a straight row, how does he proceed?

Ex. 18. From a point on a line which is not an end point (as the point O), how many directions are indicated by a line?



Ex. 19. If one of these directions is east, what is the other direction?

Ex. 20. Point out two intersecting lines in the room.

Ex. 21. Point out two lines in the room which would not meet on being produced.

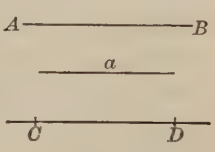
Ex. 22. Which of the capital letters of the alphabet are formed by straight lines? Curved lines? Broken lines? Curved and straight lines combined?

12. A **rectilinear figure** is a figure composed only of straight lines. A **curvilinear figure** is a figure composed only of curved lines. A **mixtilinear figure** is a figure containing both straight and curved lines.

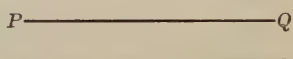
13. Kinds of straight lines.—A straight line may be definite or indefinite in length.

The line of definite length is sometimes termed a **line-segment** or, more briefly, a **segment.**

14. Naming a straight line.—A straight line is named by naming two of its points; as the line AB (a line-segment), or the line CD (indefinite in length). A segment may also be denoted by a single letter, usually small; as the line a.



15. Parallel lines are lines in a flat surface which will not meet however far they are produced; as the lines PQ and RS.

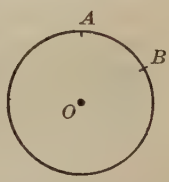


Hence two non-parallel lines will meet if produced far enough.

16. A **circle** is a closed curve all points of which lie in the same flat surface and are equidistant from a point called the **center.**

An **arc** is any portion of a circle; as AB.

A **radius** of a circle is a line drawn from the center to any point on the circle.



17. Compasses are used to draw circles, and also to mark off and compare different line-segments.

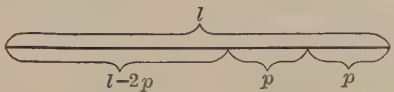
EXERCISES: GROUP 3

Ex. 1. Draw a line and denote it by l. Also draw two much shorter lines and denote them by m and p, respectively.

Ex. 2. By use of the compasses, on l mark off a part equal to m.

Ex. 3. Construct a line equal to $l-2p$. To $l-m-p$.

The following is a convenient method of indicating the work of constructing $l-2p$:



Ex. 4. Construct a line equal to $l+m-p$. To $l+p-2m$.

Ex. 5. Draw a line and on it mark off three equal segments in succession.

Ex. 6. Draw a line three times as long as a given line.

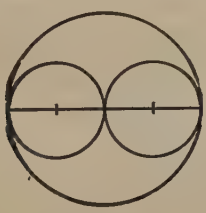
Ex. 7. Draw a circle with a radius of three fourths of an inch.

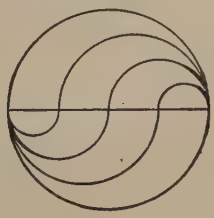
Ex. 8. Draw two circles having the same center but different radii.

Ex. 9. Draw two circles which intersect.

Ex. 10. Draw two circles which do not intersect.

Ex. 11. By use of ruler and compasses, construct figures like the following:





Ex. 12. By constructing a figure like the adjoining and erasing the proper lines, construct a six-pointed star.

Ex. 13. Draw a line, and with each end of the line as a center draw circles which will intersect.

Ex. 14. By use of the ruler and compasses, copy the following figures:

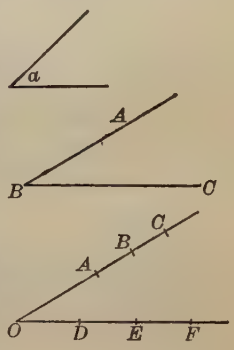
ANGLES

18. An **angle** is the amount of opening between two straight lines which meet in a point.

The **sides** of an angle are the lines whose intersection forms the angle. The **vertex** of an angle is the point in which the sides intersect. Thus, in the middle figure below, the sides of the angle are BA and BC, and the vertex is B.

19. Naming an angle.—(1) The most convenient way of naming an angle is to place a letter or figure inside the angle and near the vertex; as the angle a.

(2) The most precise way is to use three letters: one for a point on each side of the angle, with the letter at the vertex between these two letters; as the angle ABC.

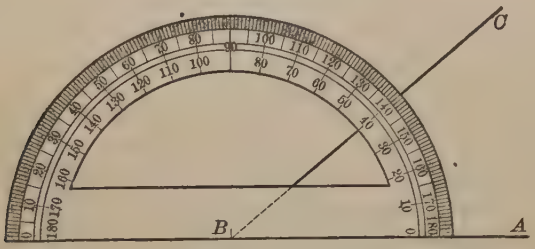


Since the size of an angle is independent of the length of its sides, the points named on its sides may be taken at any place on its sides. Thus the angles AOD, BOD, BOE, AOF are all the same angle.

(3) In case there is but one angle at a given vertex, the letter at the vertex alone may be used to denote the angle; as the angle O in the last figure.

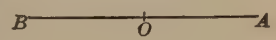
20. Unit of angle.—The customary unit of angle is the **degree** (°). If a circle is divided into 360 equal parts, and the ends of one of these parts are joined with the center by two straight lines, the angle formed at the center is 1°.

21. A **protractor** is a convenient instrument for measuring angles. It is a half circle with its rim divided into 180 equal parts, called **degrees of arc.** The center of the half circle is indicated by a dot or dash (see the mark at B).



To measure an angle ABC, place the protractor over the angle so that the center of the protractor is directly over the vertex of the angle and the zero mark on the scale is over one side (or a side produced) of the angle, as BA. The point where the other side, BC, of the angle ABC crosses the scale indicates the number of degrees in the angle. (On the diagram, this angle is 40°.)

22. A **straight angle** is an angle whose sides lie in the same straight line and extend in opposite directions from the vertex; as the angle AOB.



23. A **right angle** is half of a straight angle. Thus, if the angles x and y are each half of the straight angle BQA, x and y are right angles.

A right angle is sometimes defined as one of two equal angles formed by one straight line meeting another line.

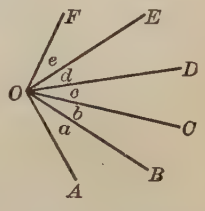
24. A **perpendicular** is a line that makes a right angle with a given line. This perpendicular is also called the **distance** from the point to the line. The point where the perpendicular meets the line is the **foot** of the perpendicular.

Thus if PQA is a right angle, PQ is perpendicular to BA, and is the distance from P to BA. Q is the foot of PQ.

EXERCISES: GROUP 4

Ex. 1. Draw two lines that intersect. Measure the four angles formed by the lines. Find the sum of any two adjacent angles.

Ex. 2. Draw a figure similar to the adjoining one, but having longer sides to the angles. Measure each of the angles a, b, c, d, and e.



Ex. 3. Compute the number of degrees in $\angle AOD$. In $\angle COF$. In $\angle AOE$.

Ex. 4. Verify your answers to Ex. 3 by measuring the angles named in it.

Ex. 5. By use of the protractor, construct angles of 30°, 60°, 90°, 15°, 45°, 120°, 135°, 180°.

Ex. 6. From a point P in a given line RS, draw a line PM making the $\angle MPS$ equal to 50°. Also measure $\angle MPR$. Find the sum of the two angles.

Ex. 7. On the diagram of Ex. 6, draw PN making $\angle NPR$ equal to 90°. Measure $\angle NPS$.

Ex. 8. Draw an angle of 60°. How many points on the scale of the protractor between the sides of this angle are marked 30°? Hence how many bisectors can an angle of 60° have?

Ex. 9. How many lines can be drawn which will bisect any given angle?

Ex. 10. Bisect the angles MPS and MPR in Ex. 6. Measure the angle formed by the two lines which bisect these angles.

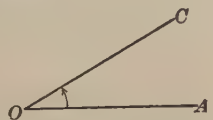
Ex. 11. Draw any two angles and then construct an angle equal to their sum.

Ex. 12. Draw two unequal angles and then construct an angle equal to their difference.

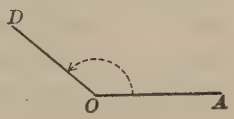
Ex. 13. How many points on the scale of the protractor are marked 90°?

Ex. 14. How many lines can be drawn perpendicular to a given line at the same point?

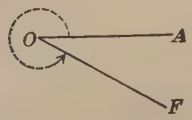
25. An **acute angle** is an angle less than a right angle; as the angle AOC.



26. An **obtuse angle** is an angle greater than a right angle but less than a straight angle; as the angle AOD.



27. A **reflex angle** is an angle greater than a straight angle, but less than two straight angles; as the angle AOF.

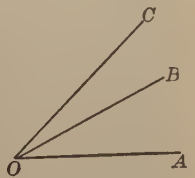


In this book, angles larger than a straight angle are not considered unless special mention is made of them.

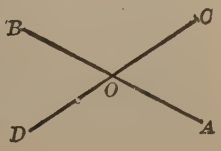
28. An **oblique angle** is an angle which is neither a right angle nor a straight angle. Hence "oblique angle" is a general term for acute, obtuse, and reflex angles.

An **oblique line** is a line which makes an oblique angle with another given line.

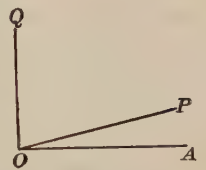
29. Two **adjacent angles** are two angles which have a common vertex and a common side between them; as the angles AOB and BOC.



30. **Vertical** (or **opposite**) **angles** are angles which have a common vertex and the sides of one angle the prolongations of the sides of the other angle; as the angles AOC and BOD.

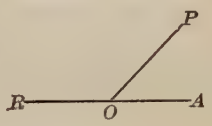


31. Complementary angles are two angles whose sum equals a right angle; as the angles AOP and POQ.



Hence the **complement of an angle** is the difference between that angle and one right angle.

32. Supplementary angles are two angles whose sum equals a straight angle (or two right angles); as the angles AOP and POR.



Hence the **supplement of an angle** is the difference between that angle and a straight angle.

EXERCISES: GROUP 5

Ex. 1. Draw an acute angle. An obtuse angle. A reflex angle.

Ex. 2. Draw two adjacent angles. Two vertical angles.

Ex. 3. How many degrees are there in the complement of an angle of 43°? In its supplement?

Ex. 4. Find the complement of 57° 19′. Of 62° 23′ 43″. Find the supplement of each of these angles.

Ex. 5. Draw an angle of 25°. What is the simplest way to construct the complement of this angle? The supplement?

Ex. 6. Is $\frac{5}{9}$ of a straight angle acute or obtuse?

Ex. 7. The dial of a clock is divided into sixty equal parts. On such a dial, how many degrees are there between two successive points of division?

Ex. 8. Find the number of degrees in the angle made by the hour hand and the minute hand of a clock at two o'clock. At three o'clock. At five o'clock.

Ex. 9. Find the number of degrees in the angle made by the hands of a clock at 1:30 o'clock. At 2:15. At 8:45.

Ex. 10. How long does it take the minute hand of a clock to turn through an angle of 60°? Of 50°? Of 240°? How long does it take the hour hand to turn through these angles?

Ex. 11. When a wheel makes $2\frac{1}{7}$ revolutions, through how many degrees does one of the spokes turn?

Ex. 12. If a spoke of a wheel turns through an angle of 144°, what fraction of a revolution does the wheel make?

Ex. 13. A pair of scales for weighing mail has a pointer moving over one half of a circular dial. If the capacity of the scales is six pounds, through how many degrees would the pointer move to indicate a weight of one pound? Three pounds? Twelve ounces? Eight ounces? One ounce?

Ex. 14. What weight will cause the pointer on the scales (Ex. 13) to move through an angle of 30°? $11\frac{1}{4}$°? 45°? 70°? 3°? 180°?

Ex. 15. What kind of angle is the supplement of an obtuse angle? Of an acute angle? Of a right angle?

Ex. 16. Draw two angles which have a common side but which are not adjacent.

Ex. 17. Draw two supplementary adjacent angles. Also two supplementary angles that are not adjacent. Also two adjacent angles that are not supplementary.

Ex. 18. The sum of a right angle and an acute angle is what kind of angle? Their difference is what kind of angle?

Ex. 19. The sum of an obtuse angle and a right angle is what kind of angle? Their difference is what kind?

Ex. 20. What kind of angle is equal to its supplement? Greater than its supplement? Less than its supplement?

Ex. 21. The difference between the supplement and the complement of any given angle is what kind of angle?

Ex. 22. What inference can you make in regard to the relative size of the angles r and t from each of the following statements?

(*a*) $\angle r$ is the comp. of $\angle s$; also $\angle s$ is the comp. of $\angle t$.
(*b*) $\angle r$ is the sup. of $\angle s$; $\angle s$ is the sup. of $\angle t$.
(*c*) $\angle r$ is the comp. of $\angle s$; $\angle s$ is the sup. of $\angle t$.
(*d*) $\angle r$ is the sup. of $\angle s$; $\angle s$ is the comp. of $\angle t$.

Ex. 23. State the inference which can be made in regard to the comparative size of angles m and p in each of the following statements:

(*a*) The comp. of $\angle m$ is less than the comp. of $\angle p$.
(*b*) The sup. of $\angle m$ equals the sup. of $\angle p$.
(*c*) The sup. of $\angle m$ is greater than the sup. of $\angle p$.
(*d*) The comp. of $\angle m$ equals the sup. of $\angle p$.

Ex. 24. How many degrees are there in an angle which equals twice its complement?

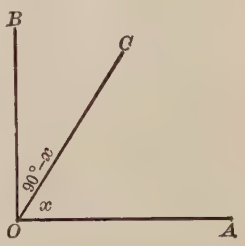
[Sug. Let $\angle AOB$ be a right angle and $\angle AOC$ the required angle.

Let $\angle AOC = x$.

Then $\angle COB = 90° - x$.

Also $x = 2(90° - x)$, etc.]

Ex. 25. How many degrees are there in an angle which equals one third of its supplement?



Ex. 26. How many degrees are there in an angle which equals four fifths of its supplement?

Ex. 27. How many degrees are there in an angle which exceeds its complement by 12°? Its supplement by 15° 30′?

Ex. 28. How many degrees are there in the angle whose complement and supplement are together equal to 126°?

Ex. 29. How many degrees are there in an angle whose supplement equals four times its complement?

Ex. 30. Make up and solve an example similar to Ex. 26. To Ex. 27.

SURFACES

33. A **plane** or **flat surface** is a surface on which a straight edge will fit in any position.

34. A **plane figure** is a figure such that all its points lie in the same plane.

35. A **curved surface** is a surface no part of which is plane.

36. Plane geometry is that branch of geometry which treats of plane figures.

37. Solid geometry is that branch of geometry which treats of figures not all points of which are in the same plane.

EXERCISES: GROUP 6

Ex. 1. What kind of surface is the floor of a room? The surface of a baseball? The surface of an egg? The surface of a hemisphere?

Ex. 2. Can you apply your ruler to a stovepipe in such a way that every point of the edge will be in contact with the pipe? Will the ruler be in contact throughout its length when applied to the pipe in any direction?

Ex. 3. Can you apply your ruler to a ball in such a way that the entire edge of the ruler will be in contact with the surface of the ball?

Ex. 4. If you should make a tennis court, how could you, by use of a plank with a straight edge, test the surface of the court so as to make sure that it was flat?

Ex. 5. If two points of a straight line are on a flat surface, will the whole line lie on the surface?

Ex. 6. Mention four flat surfaces.

Ex. 7. Mention four surfaces which are not flat.

PRIMARY RELATIONS OF GEOMETRIC MAGNITUDES; AXIOMS

38. Certain **primary relations of geometric objects** have already been given in the definitions used for geometric objects. We now proceed to investigate more generally and systematically the relations of geometric magnitudes.

39. An **axiom** is a truth accepted as requiring no demonstration.

40. Two kinds of axioms are used in geometry:

1. **General axioms,** or axioms which apply to other kinds of quantity as well as to geometric magnitudes; for instance, to numbers, forces, masses.

2. **Geometric axioms,** or axioms which apply to geometric magnitudes alone.

41. The **general axioms** may be stated as follows:

Axiom 1. *Things which are equal to the same thing, or to equal things, are equal to each other.*

AXIOM 2. *If equals are added to equals, the sums are equal.*

AXIOM 3. *If equals are subtracted from equals, the remainders are equal.*

AXIOM 4. *Doubles of equals are equal; or, in general, if equals are multiplied by equals, the products are equal.*

AXIOM 5. *Halves of equals are equal; or, in general, if equals are divided by equals, the quotients are equal.*

AXIOM 6. *Like powers or like roots of equals are equal.*

AXIOM 7. *The whole is equal to the sum of its parts.*

AXIOM 8. *The whole is greater than any of its parts.*

AXIOM 9. *A quantity may be substituted for its equal in any process.*

42. Axioms as fundamental instruments of efficiency.— The axioms given above seem so obvious that the student at first is not likely to realize their value. This value may be illustrated as follows:

If the distance from Washington to Philadelphia is known, and also the distance from Philadelphia to New York, the distance from Washington to New York may be obtained by adding together the two distances named; for, by Axiom 7, the whole is equal to the sum of its parts. Thus the labor of actually measuring the distance from Washington to New York is saved.

Again, if the height of a boy in Paris is measured, and the height of a boy in New York is also measured, and the result of the two measurements is the same, we know that the boys are of the same height, without the labor and cost of bringing the boys together and comparing their heights directly; for, by Axiom 1, things which are equal to the same thing are equal to each other.

Thus the general axioms are to be considered not merely as fundamental equivalences, but also as fundamental instruments of efficiency. For many purposes, the latter point of view is more important than the former.

43. The **geometric axioms** may be stated as follows:

GEOMETRIC AXIOM 1. *Through or connecting two given points, only one straight line can be drawn.*

GEOMETRIC AXIOM 2. *A geometric figure may be freely moved in space without any change in form or size.*

This axiom is equivalent to regarding space as *uniform;* that is, as having the same properties in all its parts.

GEOMETRIC AXIOM 3. *Through a given point outside a given straight line, one straight line and only one can be drawn parallel to the given line.*

GEOMETRIC AXIOM 4. By § 7, *geometric figures which can be made to coincide are congruent.*

44. Efficiency value of the geometric axioms.—The efficiency value of the first geometric axiom is illustrated by the fact that it enables us to determine a straight line of unlimited length when only two of its points are given. By the second axiom, the knowledge which we have of one geometric object may be transferred to another like object, however distant in space. The value of the third axiom lies partly in the inclusion or limitation which it gives, and partly in the power of transfer.

45. A **geometric postulate** is a construction of a geometric figure which, without proof, is admitted as possible.

46. The **geometric postulates** may be stated as follows:

POSTULATE 1. *Through or connecting any two points, a straight line may be drawn.*

POSTULATE 2. *A straight line may be extended indefinitely, or it may be limited at any point.*

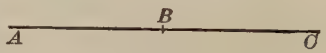
POSTULATE 3. *A circle may be described about any given point as center, and with any given radius.*

These postulates limit the student to the use of the straight-edged ruler and the compasses in constructing figures in geometry. One of the objects of the study of geometry is to discover what geometric figures can be constructed by a combination of the elementary constructions allowed in the postulates; that is, by the use of the two simplest drawing instruments.

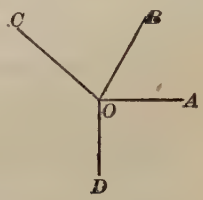
47. Logical postulates.—Besides the postulates which are used in the actual construction of figures, there are certain other postulates which are used only in the processes of reasoning. Thus, for purposes of reasoning, a given angle may be regarded as divided into any convenient number of equal parts. Whether it is possible actually thus to divide this angle on paper by use of the ruler and compasses, is another question.

EXERCISES: GROUP 7

Ex. 1. In the figure, measure AB; then measure BC. Now find AC without measuring it. What axiom have you used?



Ex. 2. If $\angle AOB = 60°$, $\angle BOC = 80°$, and $\angle COD = 130°$, find $\angle AOC$ and BOD (reflex) without measuring them. What axiom have you used?



Ex. 3. The following is a numerical illustration of the meaning of Ax. 2:

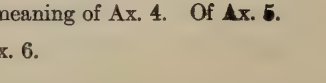
$$\begin{array}{r} 7 = 7 \\ +2 = +2 \\ \hline 9 = 9 \end{array}$$

Give a similar illustration of the meaning of Ax. 3.

Ex. 4. Give an illustration of the meaning of Ax. 4. Of Ax. 5.

Ex. 5. Illustrate the meaning of Ax. 6.

Ex. 6. If $AB = EF$,
$AD = 3AB$,
and $EH = 3EF$,
what axiom justifies us in saying that $AD = EH$?



Ex. 7. If $LM=NO$, what axiom justifies us in saying that $LN=MO$?

Ex. 8. By use of a diagram similar to that used in Ex. 7, give an illustration of Ax. 3.

Ex. 9. Give an illustration of the utility of Ax. 1 similar to that given in § 42.

Ex. 10. Give an illustration of the utility of Ax. 7.

State the axiom used in making each of the following inferences:

Ex. 11. If $a=b$ and $b=c$, then $a=c$.

Ex. 12. If $a=b$ and $c=d$, then $a+c=b+d$.

Ex. 13. If $a+b=c$ and $b=d$, then $a+d=c$.

Ex. 14. Fold a piece of paper of irregular outline so as to form a right angle by the creases. What geometric principles have you used?

DEMONSTRATION OF GEOMETRIC RELATIONS

48. A **geometric proof,** or **demonstration,** is a course of reasoning by which a relation between geometric objects is established.

49. A **geometric theorem** is a statement of a truth concerning geometric objects which requires demonstration.

Ex. The sum of the angles of a triangle equals two right angles.

50. A **geometric problem** is a statement of the construction of a geometric figure, which is required to be made.

Ex. On a given line, to construct a triangle containing three equal angles.

51. A **proposition** is a general term for either a theorem or a problem. Thus propositions are subdivided into two classes: (1) Theorems; (2) Problems.

52. Immediate inference is of two kinds:

1. Changing the point of view in a given statement.

Thus the statement, "Two straight lines drawn through two given points must coincide," may be changed to "Two straight lines cannot inclose a space."

2. Reasoning which involves but a single step.

Ex. All straight angles are equal;
$\therefore$ All right angles are equal. (Ax. 5.)

53. A **corollary** is a truth obtained by immediate inference from another truth just stated or proved, or one whose proof may be stated briefly and informally.

54. Hypothesis and conclusion.—A proposition consists of two parts:

(1) The **hypothesis,** or that which is known or granted.

(2) The **conclusion,** or that which is to be proved or constructed.

Thus, in the proposition, "If two straight lines are perpendicular to the same line, they are parallel," the hypothesis is that two given lines are perpendicular to another given line. The conclusion is that the two given lines are parallel.

PROPERTIES OF LINES INFERRED IMMEDIATELY

55. *If two straight lines have two points in common, the lines coincide throughout their whole extent.* (§ 43, Geom. Ax. 1.)

Hence *two straight lines can intersect in but one point.*

56. *If two straight lines coincide in part, they coincide throughout.*

57. *Only one straight line can be drawn connecting two given points.*

58. *Two straight lines cannot inclose a surface.*

59. *A given straight line (segment) can be divided into two equal parts at but one point.*

For (by Ax. 5) halves of equals (or of the same thing) are equal.

PROPERTIES OF ANGLES INFERRED IMMEDIATELY

60. *All straight angles are equal.*

61. *A given angle can be divided into two equal angles by but one line.*

For (by Ax. 5) halves of the same magnitude are equal.

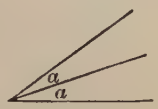


FIG. 1

62. Hence *at a given point in a straight line but one perpendicular can be erected to the line.*

63. *All right angles are equal.*

For all straight angles are equal (§ 60), and halves of equals are equal. (Ax. 5.)

64. *The sum of the two adjacent angles formed by one straight line meeting another straight line equals two right angles.*

For the angles formed are supplementary adjacent angles. (§§ 29, 32.)

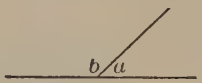


FIG. 2

65. *If two adjacent angles are together equal to a straight angle (or two right angles), their exterior sides form a straight line.*

For their exterior sides form a straight angle, and hence must lie in a straight line. (§ 22.)

66. *Complements of the same angle or of two equal angles are equal.* (§ 68 and Ax. 3.) *Supplements of the same angle or of two equal angles are equal.* (§ 60 and Ax. 3.)

67. *The sum of all the angles about a point equals four right angles.*

Thus $\angle a + \angle b + \angle c + \angle d + \angle e = 4$ rt. $\measuredangle$.

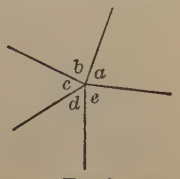


FIG. 3

68. *The sum of all the angles about a point on the same side of a straight line passing through the point is two right angles, or a straight angle.*

Thus $\angle p + \angle q + \angle r = 2$ rt. $\measuredangle$.

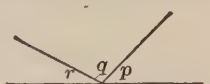


FIG. 4

EXERCISES: GROUP 8

Ex. 1. How many different straight lines are determined by three points not in the same straight line?

Ex. 2. How many straight lines are determined by four points in a plane, no three of the points being in the same straight line?

Ex. 3. How many times more efficient is a point in Ex. 2 than in Ex. 1? That is, what is the ratio of the number of lines that a point helps us to locate in the two examples?

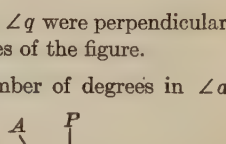
Ex. 4. How many curved lines may be drawn through two given points? How many broken lines? How many straight lines?

Ex. 5. If, in Fig. 3, p. 27, $\measuredangle a, b, c, d = 40°, 50°, 60°, 70°$, respectively, find $\angle e$.

Ex. 6. If, in Fig. 4 above, the lines forming $\angle q$ were perpendicular to each other and $\angle p = 47°$, find the other angles of the figure.

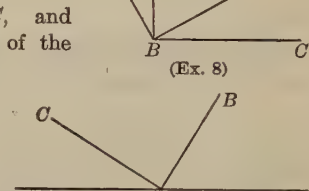
Ex. 7. By use of a protractor, find the number of degrees in $\angle a$ of Fig. 2 on page 27. Find $\angle b$ without measuring it. Now measure $\angle b$ and compare the two results.

Ex. 8. Given $QB \perp AB$, $PB \perp BC$, and $\angle ABC = 130°$; find the other angles of the figure.



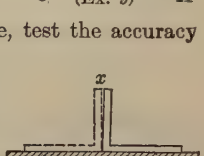
(Ex. 8)

Ex. 9. If $\angle AOC = 142°$ and $\angle BOD = 120°$, find $\angle BOC$.



(Ex. 9)

Ex. 10. By use of a board with a straight edge, test the accuracy of the outside angle of a carpenter's square by the method indicated in the diagram. How, then, would you test the accuracy of the inside angle of the square? What geometric principle have you used in each case?

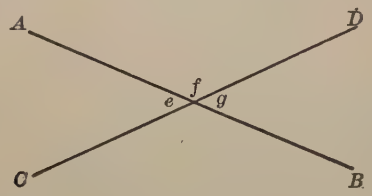


BOOK ONE

RECTILINEAR FIGURES

Proposition I. Theorem

69. *If two straight lines intersect each other, the vertical angles are equal.*



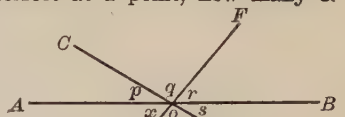
Given the straight lines AB and CD intersecting and forming the vertical $\measuredangle e$ and g.

To prove $\angle e = \angle g$.

Steps	Reasons
Proof. 1. $\angle e$ is the supplement of $\angle f$.	1. The supplement of an $\angle$ is the difference between the $\angle$ and a straight $\angle$. (§ 32.)
2. $\angle g$ is the supplement of $\angle f$.	2. Same reason as 1.
3. $\therefore \angle e = \angle g$.	3. Supplements of the same $\angle$ are equal. (§ 66.)
	Q.E.D.

Ex. 1. If one of the angles formed by two intersecting lines is 70°, find the other three angles of intersection without measuring them.

Ex. 2. If three straight lines intersect at a point, how many of the angles formed must be measured in order to determine all the angles?



Three straight lines intersect at a point O and form the angles p, q, r, s, t, x, as indicated on the diagram.

Ex. 3. If $\angle q = 110°$ and $\angle r = 47°$, find $\angle AOD$.

Ex. 4. If $\angle AOF = 153°$ and $\angle t = 107°$, find $\angle s$.

Ex. 5. If $\angle AOF = 153°$ and $\angle COB = 165°$, find $\angle t$.

Ex. 6. If $\angle p = 24°$ and $\angle q = 3\angle r$, find $\angle x$.

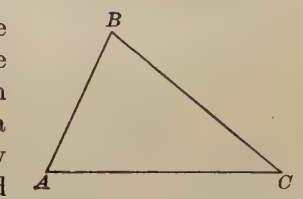
Ex. 7. Construct a diagram in which four straight lines intersect at a point, and make up and solve three examples concerning the diagram, similar to Exs. 3–6 above.

TRIANGLES

70. A **polygon** is a closed broken line.

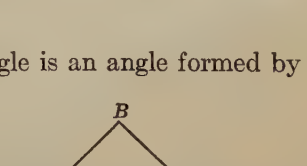
Sometimes the portion of the plane inclosed by a closed broken line is termed a polygon.

A **triangle** is a polygon of three sides; as the triangle ABC.

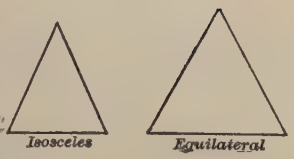


71. The **sides** of a triangle are the lines which bound it. The **perimeter** of a triangle is the sum of the sides. The **angles** of a triangle are the angles formed by the sides; as the angles A, B, and C. The **vertices** of a triangle are the vertices of the angles of the triangle.

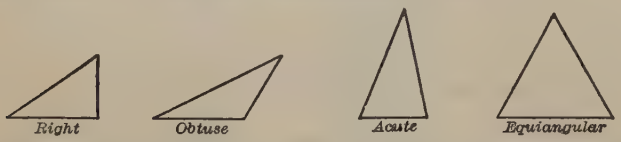
72. An **exterior angle** of a triangle is an angle formed by one side and by another side produced; as the angle BCD. With reference to the angle BCD, the angles A and B are termed the **opposite interior angles.**



73. An **isosceles triangle** is one in which at least two sides are equal. An **equilateral triangle** is one in which all three sides are equal.



74. A **right triangle** is a triangle *one* of whose angles is a right angle. An **obtuse triangle** is a triangle *one* of whose angles is an obtuse angle. An **acute triangle** is a triangle *all* of whose angles are acute angles. An **equiangular triangle** is one in which all the angles are equal.



75. The **base** of a triangle is the side upon which the triangle is supposed to stand; as AB in Figs. 1 and 2. The angle opposite the base is called the **vertex angle**; as angle ACB. The **vertex** of a triangle is the vertex of the vertex angle of the triangle.

Any side of a triangle may be considered as the base.

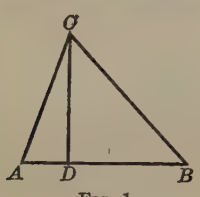


FIG. 1

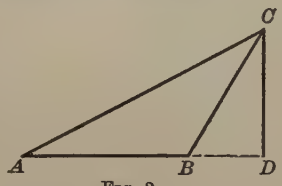


FIG. 2

The **altitude of a triangle** is the perpendicular from the vertex to the base, or to the base extended; as CD in either Fig. 1 or Fig. 2.

76. In an **isosceles** triangle, the **legs** (sometimes termed "sides") are the equal sides, and the **base** is the remaining side.

77. In a **right** triangle, the **hypotenuse** is the side opposite the right angle, and the **legs** (or "sides") are the sides adjacent to the right angle.

78. Congruent triangles are triangles which may be made to coincide.

Lines which may be made to coincide are termed equal (not congruent) lines. This practice is due to the simplicity of a line as compared with a triangle. Similarly we speak of equal, not congruent, angles.

79. Property of a triangle immediately inferred.—*The sum of any two sides of a triangle is greater than the third side.* For a straight line is the shortest line joining two points. (§ 9.)

80. A **quadrilateral** is a polygon of four sides.

A **square** is a quadrilateral all of whose sides are equal and all of whose angles are right angles.

Ex. 1. Point out the hypothesis and conclusion in Prop. I (p. 29).

Ex. 2. Find the angle whose complement is 18°. The angle whose supplement is 76°.

Ex. 3. If the complement of an angle is known, what is the shortest way of finding the supplement of the angle? If the supplement is known, what is the shortest way of finding the complement?

Ex. 4. Draw three straight lines so that they intersect in three points. In two points. In one point.

Ex. 5. If one side of an equilateral triangle is 4 inches, what is the perimeter of the triangle?

Ex. 6. Is it possible to form a triangle whose sides are 6, 9, and 17 inches? Try to do this with the compasses and ruler.

Ex. 7. Is it possible to form a triangle in which one side is 10 inches and the difference of the other two sides is 12 inches?

THE BRIDGE-BUILDER USES GEOMETRY

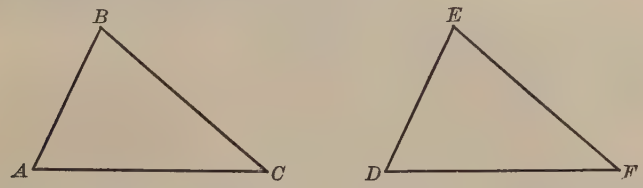
Notice the use, in constructing this bridge across the Hudson River, of triangles and circular arches in steel structure where strength is needed. Do you know why these geometric figures are chosen? (*Ewing Galloway*)

ANOTHER GEOMETRIC BRIDGE

This completed bridge in Germany shows the same principles of construction as the Hudson River Bridge. Greater prominence is given the arch, but the triangle is still used for bracing. What geometric figures can you find in the bridge? (*Ewing Galloway*)

Proposition II. Theorem

81. *If two triangles have two sides and the included angle of one respectively equal to two sides and the included angle of the other, the triangles are congruent.*



Given the $\triangle ABC$ and DEF in which $AB=DE$, $AC=DF$, and $\angle A=\angle D$.

To prove $\triangle ABC\cong\triangle DEF$.

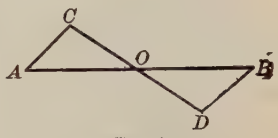
Steps	Reasons
Proof. 1. Place the $\triangle ABC$ upon the $\triangle DEF$ so that side AC coincides with its equal DF, and AB falls on the same side of DF as DE.	1. A geometric object may be freely moved in space without any change in size or form. (Geom. Ax. 2, § 43.)
2. Then the side AB will take the direction of DE.	2. $\angle A=\angle D$ by hypothesis.
3. The point B will fall on E.	3. $AB=DE$ by hypothesis.
4. The side BC will coincide with EF.	4. Only one straight line can be drawn connecting two given points. (§ 57.)
5. $\triangle ABC\cong\triangle DEF$.	5. Geometric figures which can be made to coincide are congruent. (Geom. Ax. 4, § 43.)

Q.E.D.

82. Corresponding sides and angles.—If two triangles are congruent, a pair of angles in them which may be made to coincide, as $\measuredangle A$ and D above, are said to be **corresponding.** Corresponding sides are sides opposite corresponding angles, as BC and EF.

EXERCISES: GROUP 9

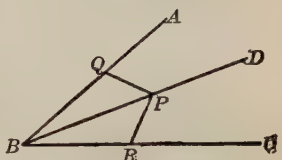
Ex. 1 Given lines AB and CD bisecting each other at O. Prove $\triangle ACO \cong \triangle OBD$.



(Ex. 1)

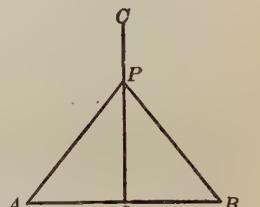
Ex. 2. Construct a square $ABCD$. Let M be the midpoint of the side AD. Draw the lines BM and CM. Prove $\triangle ABM \cong \triangle MCD$.

Ex. 3. Given $\angle ABC$ bisected by BD, P any point in BD, $BQ = BR$. Prove $\triangle BPQ \cong \triangle BRP$.



(Ex. 3)

Ex. 4. Given O the midpoint of the line AB, $CO \perp AB$, P any point in CO. Prove $\triangle APO \cong \triangle OPB$.

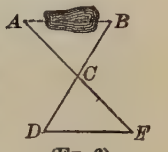


(Ex. 4)

Ex. 5. Construct a figure and give proof for the following:

Given $\triangle ABC$ with side AB = side BC; AB produced through B to Q, and CB produced through B to P; also $PB = BQ$. Prove $\triangle APB \cong \triangle BQC$.

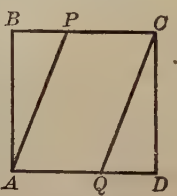
Ex. 6. To determine the distance AB, take a convenient position C, measure AC, and produce AC to F, making $CF = AC$. Also measure BC and produce BC to D, making $CD = BC$. Prove $\triangle DCF \cong \triangle ACB$. If we now measure DF and find it to be 217 yd., how long is AB?



(Ex. 6)

Ex. 7. Construct a square $ABCD$. Let M be the midpoint of the side CD. Let P be a point on AD, and Q a point on BC such that $PD = QC$. Prove $\triangle QCM \cong \triangle PMD$.

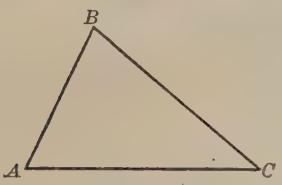
Ex. 8. Given the square $ABCD$, with P a point on BC and Q a point on AD such that $PC = AQ$. Prove $\triangle ABP \cong \triangle QCD$.

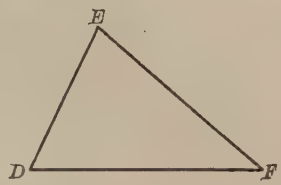


(Ex. 8)

Proposition III. Theorem

83. *If two triangles have a side and the two adjacent angles of one respectively equal to a side and the two adjacent angles of the other, the triangles are congruent.*





Given the $\triangle ABC$ and DEF in which $\angle A = \angle D$, $\angle C = \angle F$, and $AC = DF$.

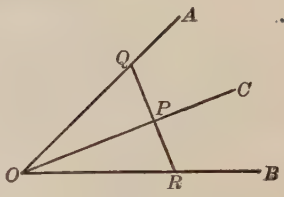
To prove $\triangle ABC \cong \triangle DEF$.

Steps	Reasons
Proof. 1. Place the $\triangle ABC$ upon the $\triangle DEF$ so that AC coincides with its equal DF, and the point B falls on the same side of DF as E.	1. A geometric object may be freely moved in space without any change in size or form. (Geom. Ax. 2, § 43.)
2. Then AB will take the direction of DE.	2. $\angle A = \angle D$ by hypothesis.
3. Also CB will take the direction of FE.	3. $\angle C = \angle F$ by hypothesis.
4. Point B will fall on E.	4. Two straight lines can intersect in but one point. (§ 55.)
5. $\triangle ABC \cong \triangle DEF$.	5. Geometric figures which can be made to coincide are congruent. (Geom. Ax. 4, § 43.) Q.E.D.

Ex. By use of ruler and compasses construct a triangle whose sides are 2in., $1\frac{3}{4}$ in., and $1\frac{1}{2}$ in.

EXERCISES: GROUP 10

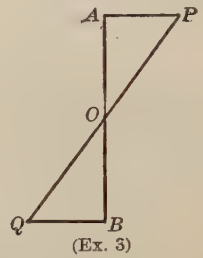
Ex. 1. Given $\angle AOB$ bisected by OC, P any point in OC, line $QPR \perp OC$. Prove $\triangle QOP \cong \triangle POR$.



(Ex. 1)

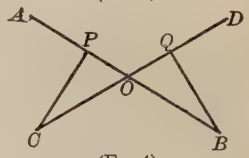
Ex. 2. Construct a square $ABCD$. Let M be the midpoint of the side AD. Let P be a point on the side AB and Q a point on CD, such that $\angle PMA = \angle QMD$. Prove $\triangle APM \cong \triangle MQD$.

Ex. 3. Given O the midpoint of line AB, POQ any straight line through point O, AP and $QB \perp AB$. Prove $\triangle OAP \cong \triangle OQB$.



(Ex. 3)

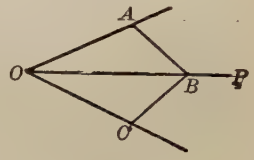
Ex. 4. Given lines AB and CD intersecting at O, $OP = OQ$, $PC \perp AB$, $QB \perp CD$. Prove $\triangle CPO \cong \triangle OQB$.



(Ex. 4)

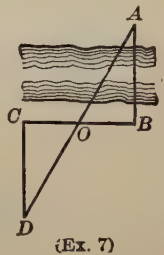
Ex. 5. Draw a line AB, having the midpoint M. Through M draw any line CD such that $\angle CAM = \angle DBM$. Prove $\triangle AMC \cong \triangle MDB$.

Ex. 6. Given $\angle AOC$ bisected by line OP, B any point in OP, $\angle ABP = \angle CBP$. Prove $\triangle OAB \cong \triangle OCB$.



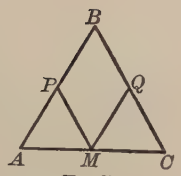
(Ex. 6)

Ex. 7. To measure the distance AB, construct the line BC perpendicular to AB, making $CO = OB$. Construct $CD \perp CB$ and meeting AO produced at D. Prove $\triangle DCO \cong \triangle OAB$. If CD is measured and found to be 137 yd., how long is AB?



(Ex. 7)

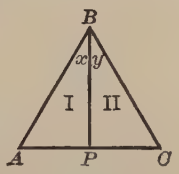
Ex. 8. Given $\triangle ABC$, M the midpoint of AC, $\angle A = \angle C$, $AP = QC$. Prove $\triangle APM \cong \triangle MQC$.



(Ex. 8)

Ex. 9. Construct a square $ABCD$ in which M, P, and Q are the midpoints of the sides AD, AB, and CD, respectively. Prove $\triangle APM \cong \triangle MQD$.

Ex. 10. Given $\triangle ABC$, $BP \perp AC$, $\angle x = \angle y$. Prove $\triangle I \cong \triangle II$.



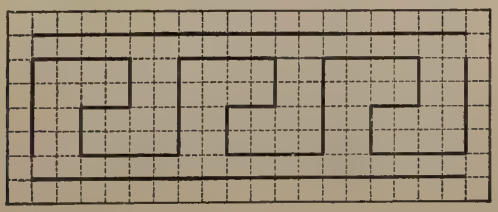
(Ex.10)

Ex. 11. Draw a square $ABCD$, with P and Q points on the sides BC and CD, respectively, such that $PC = CQ$. Prove $\triangle ABP \cong \triangle AQD$.

Ex. 12. If $PQRS$ is a square and A a point on QR and B a point on PS, such that $AR = PB$, prove $\angle QPA = \angle BRS$.

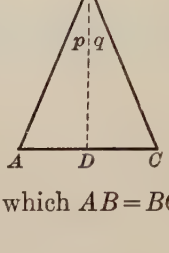
Ex. 13. DFH is a triangle in which $\angle D = \angle H$. M is the midpoint of the side DH, and A a point on the side DF and B a point on FH, such that $\angle AMD = \angle BMH$. Prove $\triangle DMA \cong \triangle MBH$.

Ex. 14. By the aid of squared paper, construct the following design, called a meander line. (It is an advantage to draw the meander line in blue or red pencil or in red ink.)



Proposition IV. Theorem

84. *If two sides of a triangle are equal, the angles opposite these sides are equal.*



Given the $\triangle ABC$ in which $AB = BC$.

To prove $\angle A = \angle C$.

Steps	Reasons
Proof. 1. Let BD be drawn so as to bisect $\angle ABC$ and meet AC in D.	1. A given $\angle$ can be divided into two equal parts by but one line. (§§ 61, 47.)
2. Then in the $\triangle ABD$ and DBC, $AB = BC$.	2. Hypothesis.
3. $BD = BD$.	3. Identity.
4. $\angle p = \angle q$.	4. Construction.
5. $\therefore \triangle ABD \cong \triangle CBD$.	5. § 81. (Let the student quote § 81 in full.)
6. $\angle A = \angle C$.	6. Corr. $\measuredangle$ of congruent $\triangle$ are $=$. Q.E.D.

85. Cor. 1. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.*

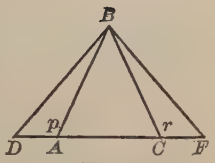
86. Cor. 2. *An equilateral triangle is equiangular.*

Ex. 1. In the diagram of Prop. IV, if $AB = 10$ in., $BC = 10$ in., and $\angle A = 67°$, how do we find the number of degrees in $\angle C$ without measuring the angle?

Ex. 2. Draw a triangle whose altitude coincides with one side.

EXERCISES: GROUP 11

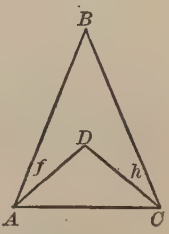
Ex. 1. Given $\triangle ABC$ with $AB=BC$, and AC produced so that $AD=CF$. Prove $\angle p=\angle r$; also $\triangle DBA\cong\triangle CBF$.



(Ex. 1)

Ex. 2. Draw a square $ABCD$. Also the line AC. Prove $\angle BAC=\angle BCA$.

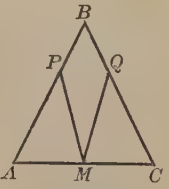
Ex. 3. Draw a square $DFHK$. Draw the diagonals DH and FK. Then prove $\triangle DFK\cong\triangle DHK$. Also prove that the diagonals are equal.



(Ex. 4)

Ex. 4. Given $\triangle ABC$ and ADC on the same base AC, with $AB=BC$, and $AD=DC$. Prove $\angle f=\angle h$.

Ex. 5. On the common base BC, construct two isosceles triangles BAC and BDC, one above BC and the other below it. Prove $\angle ABD=\angle ACD$.



(Ex. 6)

Ex. 6. Given $\triangle ABC$, with $AB=BC$, M the midpoint of AC, and $AP=QC$. Prove $\triangle APM\cong\triangle MQC$.

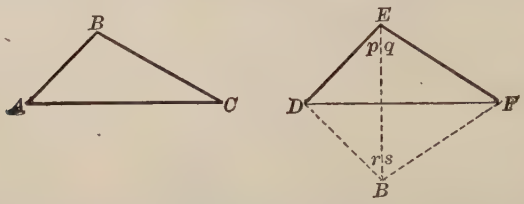
Ex. 7. ABC is a triangle in which $AB=BC$. P and Q are points on AC such that $AP=QC$. R and S are points on AB and BC, respectively, such that $AR=SC$. Draw the figure and prove $\triangle ARP\cong\triangle QSC$.

Ex. 8. DFH is an isosceles triangle in which $DF=FH$. P and Q are points on DH such that $DP=QH$. Prove $\triangle DFP\cong\triangle QFH$ and also $FP=FQ$. Also name all the pairs of equal angles which you can find on the figure.

Ex. 9. In the quadrilateral $ABCD$, $AB=AD$ and the diagonal AC bisects the angle BAD. Prove that AC also bisects the angle BCD.

Proposition V. Theorem

87. *If two triangles have the three sides of one respectively equal to the three sides of the other, the triangles are congruent.*



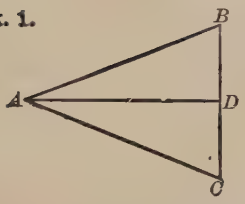
Given the △ ABC and DEF in which $AB=DE$, $BC=EF$, and $AC=DF$.

To prove $\triangle ABC \cong \triangle DEF$.

Steps	Reasons
Proof. 1. Place $\triangle ABC$ so that the longest side AC coincides with its equal DF in the $\triangle DEF$, and the vertex B falls on the opposite side of DF from E.	1. Geom. Ax. 2.
2. Draw the line EB.	2. Postulate 1, § 46.
3. Then $DE=DB$.	3. Hypothesis.
4. $\angle p=\angle r$.	4. If two sides of a △ are equal, the ∠ opposite these sides are equal. (§ 84.)
5. In like manner, in the $\triangle BEF$, $\angle q=\angle s$.	5. Same reasons as for steps 3 and 4.
6. Hence $\angle p+\angle q=\angle r+\angle s$.	6. If equals are added to equals, the sums are equal. (Ax. 2, § 41.)
7. Or $\angle DEF=\angle DBF$.	7. The whole is equal to the sum of its parts. (Ax. 7, § 41.)
8. $\therefore \triangle DBF$, or $\triangle ABC \cong \triangle DEF$.	8. § 81. **Q.E.D.**

EXERCISES: GROUP 12

Ex. 1.

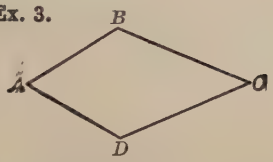


Given $\triangle ABC$ in which $AB = AC$ and D is the midpoint of BC.
Prove $\triangle ABD \cong \triangle ADC$.

Ex. 2.

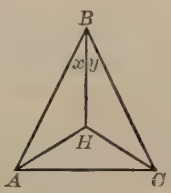
Given $DA = FB$, $DB = AF$.
Prove $\triangle DAF \cong \triangle DBF$.

Ex. 3.



Given the quadrilateral $ABCD$ in which $AB = AD$ and $BC = CD$. Draw a line connecting two of the vertices, which shall divide the figure into two congruent triangles. Give proof that the triangles formed are congruent.

Ex. 4.

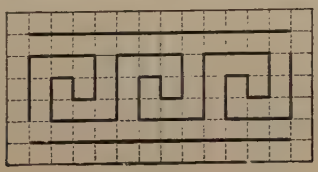


Given $\triangle ABC$, $AB = BC$, and $AH = HC$. Prove $\angle x = \angle y$.

Ex. 5. $ABCD$ is a square, with M the midpoint of the side AD. Draw the lines BM and CM, and prove $\angle BMA = \angle CMD$.

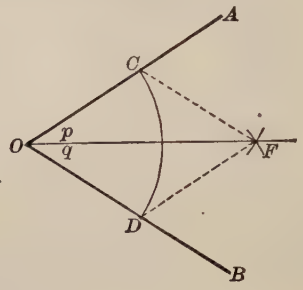
Ex. 6. AB and CD are two line-segments bisecting each other at P. Draw the figure and prove $AC = DB$.

Ex. 7. By the aid of squared paper, construct the following design:



Proposition VI. Problem

88. *To bisect a given angle.*



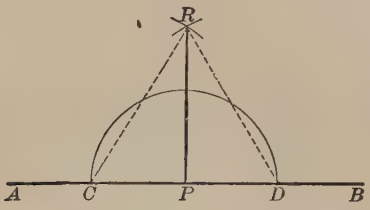
Given angle AOB.

To bisect angle AOB.

Construction. 1. With O as a center and with a radius OC, draw an arc meeting OA at C and OB at D.	1. A circle may be described about any given point as center, and with any given radius. (Post. 3, § 46.)
2. With C and D as centers and with convenient equal radii greater than $\frac{1}{2}$ the distance from C to D, describe arcs intersecting at F.	2. Post. 3.
3. Draw OF.	3. Post. 1.
Then OF bisects $\angle AOB$.	
Proof. 1. Draw CF and DF.	1. Post. 1.
2. In the $\triangle OCF$ and ODF, $OF = OF$.	2. Ident.
3. $OC = OD$.	3. Constr.
4. $CF = DF$.	4. Constr.
5. $\therefore \triangle OCF \cong \triangle ODF$.	5. If two $\triangle$ have the three sides of one respectively equal to the three sides of the other, the $\triangle$ are congruent (§ 87).
6. $\angle p = \angle q$, or $\angle AOB$ is bisected by OF.	6. Corr. $\measuredangle$ of congruent $\triangle$ are $=$. Q.E.F.

Proposition VII. Problem

89. *At a given point of a given line, to erect a perpendicular to the line.*



Given the point P in the line AB.

To construct a perpendicular to the line AB at the point P.

Construction. 1. From P as a center, with a convenient radius, describe an arc cutting off equal segments PC and PD on the line AB.	1. Post. 3.
2. From C and D as centers, with equal radii greater than PD, describe arcs intersecting at R.	2. Post. 3.
3. Draw PR.	3. Post. 1.
Then PR is the perpendicular required.	
Proof. 1. Draw the lines CR and DR.	1. Post. 1.
2. Then in the $\triangle CRP$ and PRD, $RP = RP$.	2. Ident.
3. $CP = PD$.	3. Constr.
4. $CR = RD$.	4. Constr.
5. $\therefore \triangle CRP \cong \triangle PRD$.	5. Why?
6. $\therefore \angle CPR = \angle DPR$.	6. Why?
7. $\therefore \angle CPR =$ rt. $\angle$, $RP \perp AB$.	7. §§ 23, 24. Q.E.F

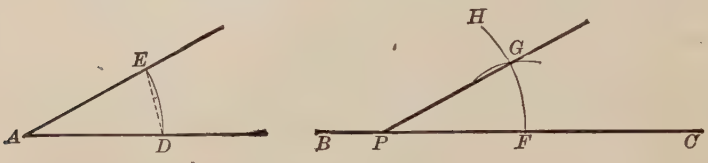
Ex. 1. Construct an equilateral triangle and bisect its angles.

Ex. 2. Construct an angle of 45°. An angle of 135°.

Ex. 3. Construct a right triangle whose legs are 1 in. and $1\frac{1}{2}$ in.

Proposition VIII. Problem

90. *To construct an angle equal to a given angle.*



Given the $\angle A$.

To construct an angle equal to $\angle A$.

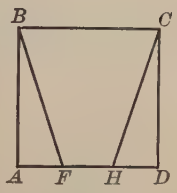
Construction. 1. With A as a center, and with any convenient radius, describe an arc meeting the sides of $\angle A$ at D and E.	1. Post. 3.
2. Draw DE.	2. Post. 1.
3. Let P be any point on a convenient line-segment BC. With P as a center and with a radius equal to AD, describe an arc HF cutting PC at F.	3. Post. 3.
4. With F as a center and with a radius equal to DE, describe an arc intersecting the arc HF at G.	4. Post. 3.
5. Draw PG.	5. Post. 1.
Then $\angle FPG$ is the angle required.	
Proof. Let the student supply the proof by drawing FG and proving the $\triangle AED$ and PGF congruent.	**Q.E.F.**

Ex. 1. At a given point in a given line, construct an angle of 45°, one of whose sides shall be the given line.

Ex. 2. Construct a triangle in which two of the sides are $1\frac{1}{4}$ in. and $1\frac{1}{2}$ in., and the angle included by them is 45°.

Ex. 3. Construct a triangle in which two of the angles are 45° and 90°, and the side included by them is $1\frac{1}{2}$ in.

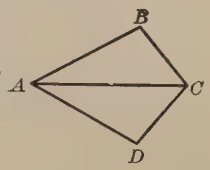
EXERCISES: GROUP 13



(Ex. 1)

Ex. 1. Given $ABCD$, a square, $AF = HD$. Prove $BF = CH$.

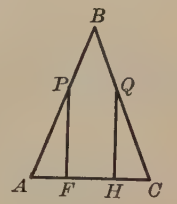
Ex. 2. Given the lines AB and CD intersecting each other at the point M so that $AM = MB$, and the lines AC and DB drawn making $\angle A = \angle B$. Prove $CM = DM$.



(Ex. 3)

Ex. 3. Given the quadrilateral $ABCD$. AC bisects $\measuredangle BAD$ and BCD. Prove $BC = CD$ and $AB = AD$.

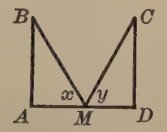
Ex. 4. $ABCD$ is a square. P is a point on BC and Q a point on CD such that $PC = CQ$. Prove $AP = AQ$.



(Ex. 5)

Ex. 5. Given $\triangle ABC$, $AB = BC$, $AF = HC$, PF and $QH \perp AC$. Prove $\angle APF = \angle CQH$.

Ex. 6. Draw a square $ABCD$. Let P, Q, R, and S be the midpoints of the sides AB, BC, CD, and DA, respectively. Prove that the four sides of the figure formed by drawing the lines PQ, QR, RS, and SP are equal.



(Ex. 7)

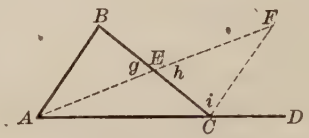
Ex. 7. Given M the midpoint of AD, AB and $CD \perp AD$, $AB = CD$. Prove $\angle x = \angle y$.

Ex. 8. $ABCD$ is a quadrilateral in which $AB = AD$ and AC bisects the angle BAD. Prove $BC = CD$. How many pairs of equal angles are there on the figure? Name them.

Ex. 9. Given the quadrilateral $ABCD$ with the line BD drawn, $AB = CD$, and $BC = AD$. Prove $\angle A = \angle C$.

Proposition IX. Theorem

91. *An exterior angle of a triangle is greater than either opposite interior angle.*



Given $\angle BCD$ an exterior $\angle$ of the $\triangle ABC$.

To prove $\angle BCD$ greater than $\angle B$ or $\angle BAC$.

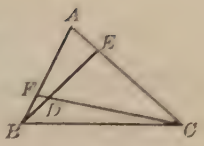
Proof. 1. Let E be the mid-point of the line BC.	1. A given straight line (segment) can be divided into two equal parts at but one point. (§§ 59, 47.)
2. Draw AE and produce it to F, making $FE=AE$. Draw FC.	2. Posts. 1 and 2.
3. Then in the $\triangle AEB$ and FEC, $BE=EC$, $AE=EF$.	3. Constr.
4. $\angle g=\angle h$.	4. Why?
5. $\therefore \triangle ABE \cong \triangle EFC$.	5. Why?
6. $\therefore \angle B=\angle i$.	6. Why?
7. But $\angle BCD$ is greater than $\angle i$.	7. The whole is greater than any of its parts. (Ax. 8.)
8. Substitute $\angle B$ for its equal $\angle i$. Then $\angle BCD$ is greater than $\angle B$.	8. A quantity may be substituted for its equal in any process. (Ax. 9.)

By producing BC through C to H, it may be proved that $\angle ACH$ (which $=\angle BCD$) is greater than $\angle BAC$. **Q.E.D.**

Ex. 1. By use of ruler and compasses construct a right triangle whose legs are 2 in. and $1\frac{1}{2}$ in., and bisect all its angles.

Ex. 2. Construct an isosceles triangle with a base $1\frac{1}{2}$ in., and legs each 2 in., and bisect its vertex angle.

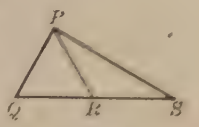
Ex. 3. In the $\angle ABC$, F is any point on AB, and E any point on AC. In the figure, what angles are less than $\angle BDC$? Than $\angle BFD$? Than $\angle FDE$? Why?



(Ex. 3)

Ex. 4. Draw a quadrilateral $ABCD$ and also its diagonals AC and BD. Let AC and BD intersect in the point O. State what angles on the figure must be less than angle AOB. Also those less than angle AOD.

Ex. 5. In the figure, $PQ = PR$, and QRS is a straight line. Prove $\angle S$ is less than $\angle Q$.



(Ex. 5)

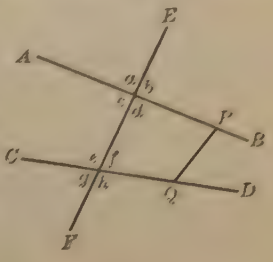
Ex. 6. ABC is a right triangle with the right angle at B. Prove that the angles A and C are both acute angles.

[Suc. Produce CB through B to H.]

92. A transversal is a line that either meets or intersects two or more other lines.

Thus, in the figure below, PQ and EF are both transversals of the lines AB and CD.

If two lines are cut by a transversal, it is convenient to give special names to the eight angles of intersection.



a, b, g, h are called **exterior angles.**

c, d, e, f are called **interior angles.**

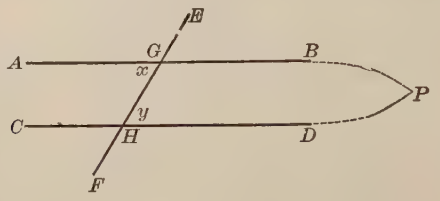
c, f form a pair of **alternate interior angles.**

b, f form a pair of **corresponding angles.**

Let the student name another pair of alternate interior angles in the above figure. Also another pair of corresponding angles. Also a pair of interior angles on the same side of the transversal.

Proposition X. Theorem

93. *If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the two lines are parallel.*



Given AB and CD, two lines cut by the transversal EF at the points G and H, respectively, making $\angle x = \angle y$.

To prove $AB \parallel CD$.

Proof. 1. If AB and CD are not $\parallel$, they must meet at some point P and thus with GH form the $\triangle GPH$.	1. Two non-parallel lines will meet if produced far enough. (§ 15.)
2. $\angle x$ would then be greater than $\angle y$.	2. An exterior $\angle$ of a $\triangle$ is greater than either opp. interior $\angle$. (§ 91.)
3. But $\angle x$ cannot be greater than $\angle y$.	3. $\angle x = \angle y$ by hyp.
4. $\therefore AB \parallel CD$.	4. § 15. For it has been proved that AB and CD cannot meet. Q.E.D.

94. Cor. *If two lines are cut by a transversal so that a pair of alternate interior angles are unequal, the two lines are not parallel.*

Ex. Prove that the line which bisects the vertex angle of an isosceles triangle is perpendicular to the base.

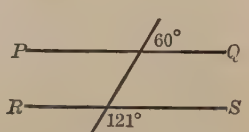
95. Methods of geometric proof.—An examination of the demonstrations of propositions given thus far shows that several different kinds of proof are used in geometry. Of these the principal kinds are as follows:

1. **Direct demonstration.**—Examples of this kind of proof are given on pp. 29 and 38.

2. **Proof by superposition,** in which two figures are proved equal by placing one of the figures upon the other and showing that the two figures must coincide. See the proofs on pp. 33 and 35.

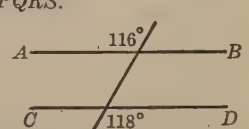
3. **Indirect demonstration,** which consists essentially in showing that a given statement is true by showing that its negative cannot be true. An illustration of this kind of proof is the demonstration which has just been given in § 93.

Ex. 1. Are the lines PQ and RS parallel? Why?



Ex. 2. $PQRS$ is a quadrilateral in which $PQ=PS$ and $QR=SR$. Prove the diagonal PR bisects two angles of $PQRS$.

Ex. 3. Are the lines AB and CD parallel? Why?



Ex. 4. In the triangle CAD, AB is the altitude drawn to the base CD. Also $BC=BD$. Prove that the triangle CAD is isosceles.

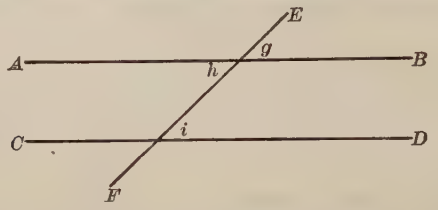
Ex. 5. The angle AOB is bisected by the line OC. P is any point on OC. Q is a point on OA, and R a point on OB, such that $OQ=OR$. Prove $\angle QPC=\angle RPC$.

Ex. 6. $ABCD$ is a square and P and Q are points on BC such that $BP=CQ$. Prove $\angle QAD=\angle PDA$.

Ex. 7. By use of ruler and compasses construct a quadrilateral $ABCD$ such that $AB=1\frac{1}{2}$ in., $BC=2$ in., $AD=1\frac{1}{2}$ in., $AC=2\frac{1}{2}$ in., and $\angle BAD=90°$.

Proposition XI. Theorem

96. *If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.*



Given the lines AB and CD cut by the transversal EF, making the corresponding $\measuredangle$ g and i equal.

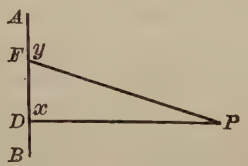
To prove $AB \parallel CD$.

Proof. 1.	$\angle h = \angle g$.	1. Why?
2.	$\angle i = \angle g$.	2. Hyp.
3.	$\therefore \angle h = \angle i$.	3. Why?
4.	$\therefore AB \parallel CD$.	4. § 93. **Q.E.D.**

97. Cor. *Two lines in the same plane perpendicular to the same line are parallel.*

Proposition XII. Theorem

98. *Not more than one perpendicular can be drawn to a given line from a given external point.*



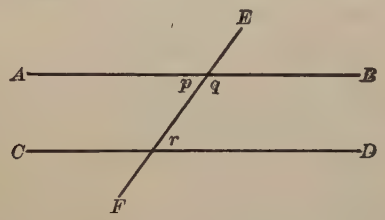
Given PD a $\perp$ and PF any other line from P to AB.

To prove that PD is the only $\perp$ from P to AB.

Proof. 1. $\angle y > \angle x$.	1. § 91.
2. $\angle x$ is a rt. $\angle$.	2. Hyp.
3. $\therefore \angle y$ is an obtuse $\angle$.	3. § 26.
4. $\therefore PF$ is not $\perp AB$; that is, PD is the only $\perp$ from P to AB.	4. § 24. **Q.E.D.**

Proposition XIII. Theorem

99. *If two lines are cut by a transversal so that two interior angles on the same side of the transversal are supplementary, the lines are parallel.*



Given the lines AB and CD cut by the transversal EF, q and r int. $\measuredangle$ on the same side of EF, and $\angle q + \angle r = 180°$.

To prove $AB \parallel CD$.

Proof. 1. $\angle p$ is the sup. of $\angle q$.	1. § 32.
2. $\angle r$ is the sup. of $\angle q$.	2. Hyp.
3. $\therefore \angle p = \angle r$.	3. Supplements of the same angle are equal. § 66.
4. $\therefore AB \parallel CD$.	4. § 93. **Q.E.D.**

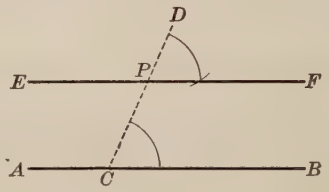
Ex. 1. On the diagram on page 46, name two pairs of alternate interior angles. Also name one pair of corresponding angles. Also one pair of interior angles on the same side of a transversal.

Ex. 2. By use of Prop. XI, give a complete proof of § 97.

Ex. 3. If, on the diagram of Prop. XIII, q exceeds r by 88° and $q = 3r$, show whether AB and CD are parallel.

Proposition XIV. Problem

100. *Through a given point, to draw a straight line parallel to a given straight line.*



Given any point P outside the line AB.

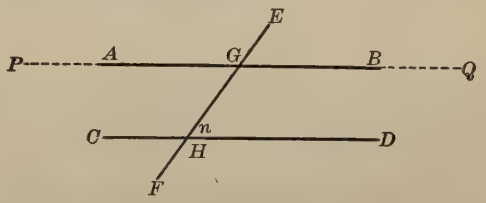
To construct a line through $P \parallel AB$.

Construction. 1. Through P draw any convenient line CD meeting AB in C.	1. Post. 1.
2. At P, and on the same side of CD with B, construct $\angle DPF$ equal to $\angle PCB$.	2. § 90.
Then EF is the line required.	
Proof. 1. $\angle DPF = \angle PCB$.	1. Constr.
2. $\therefore EF \parallel AB$.	2. § 96.
	Q.E.F.

Ex. Through each vertex of a $\triangle$, construct a line $\parallel$ opposite side.

Proposition XV. Theorem

101. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



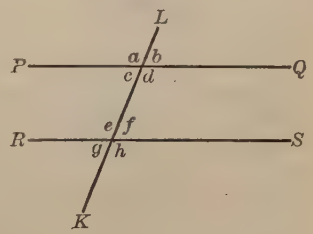
Given the $\parallel$ lines AB and CD cut by EF at the points G and H, respectively

To prove $\angle AGH = \angle n$.

Proof. 1. Through the point G draw the line PQ, making $\angle PGH = \angle n$.	1. § 90.
2. $\therefore PQ \parallel CD$.	2. § 93.
3. But $AB \parallel CD$.	3. Hyp.
4. $\therefore PQ$ and AB coincide in direction.	4. Geom. Ax. 3, § 43.
5. $\therefore \angle AGH = \angle n$.	5. $\angle AGH$ coincides with $\angle PGH$, which $= \angle n$. **Q.E.D.**

102. Cor. 1. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Thus, if $PQ \parallel RS$, $\angle b = \angle f$, since each of them $= \angle c$.



103. Cor. 2. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

Thus $\angle f$ is the supplement of $\angle d$, since $\angle f = \angle c$ (§ 101) which is the supplement of $\angle d$ (§ 32).

104. Cor. 3. *If, when two lines are cut by a transversal, the interior angles on the same side of the transversal are not supplementary, the two lines are not parallel.*

105. Cor 4. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

106. Cor. 5. *Two lines perpendicular, respectively, to two intersecting lines also intersect.*

107. Cor. 6. *Two straight lines parallel to a third straight line are parallel to each other;*

Lines parallel to parallel lines are parallel;

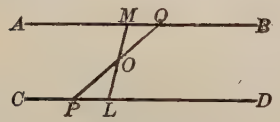
Lines perpendicular to parallel lines are parallel.

Ex. 1. If, in the diagram of § 102, $\angle b$ equals 67°, find the other seven angles in the figure without measuring them.

Ex. 2. Through a point P outside the line AB, draw three lines making $\measuredangle$ of 30°, 60°, and 45°, respectively, with AB.

EXERCISES: GROUP 14

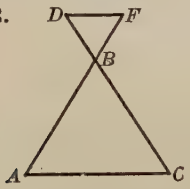
Ex. 1.



Given $AB \parallel CD$, PQ and ML straight lines, and $PO = OQ$.

Prove $\triangle POL \cong \triangle OMQ$.

Ex. 2.



Given $\triangle ABC$ in which $AB = BC$; AB produced to F; CB to D; $DF \parallel AC$.

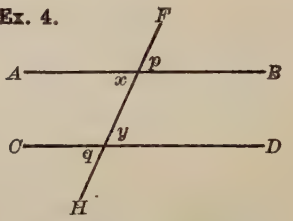
Prove $\angle D = \angle F$.

Ex. 3.

Given AB and CD = and $\parallel$.

Prove that lines AD and BC bisect each other.

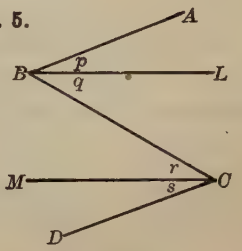
Ex. 4.



Given AB and CD intersected by transversal FH, $\angle p = \angle q$.

Prove $AB \parallel CD$.

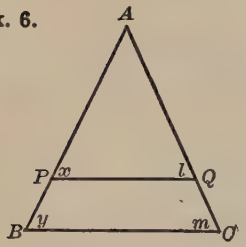
Ex. 5.



Given lines meeting at B and C so that $\angle p = \angle s$ and $\angle q = \angle r$.

Prove $AB \parallel CD$.

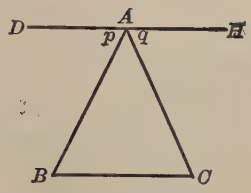
Ex. 6.



Given $\triangle ABC$ in which $AB = AC$, and $PQ \parallel BC$.

Prove $\angle x = \angle l$.

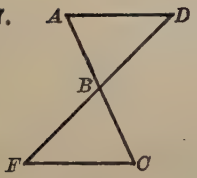
Ex. 8.



Given $\triangle ABC$ in which $AB = AC$, line $DAH \parallel BC$.

Prove $\angle p = \angle q$.

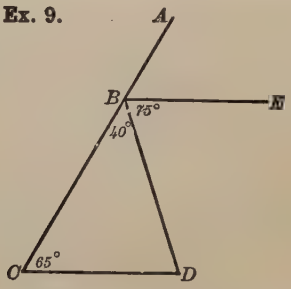
Ex. 7.



Given lines AC and DF bisecting each other at B.

Prove $AD \parallel FC$.

Ex. 9.



Given $\triangle BCD$ with exterior $\angle ABD$, $\angle C = 65°$, $\angle CBD = 40°$, $\angle DBE = 75°$.

Prove $BE \parallel CD$.

Ex. 10. On the diagram of Prop. IX (p. 46), prove that CF is parallel to AB.

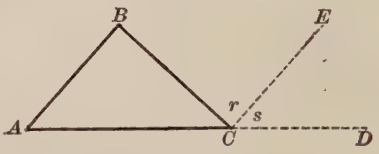
Ex. 11. In the quadrilateral $ABCD$, $BC = CD$ and BC is parallel to AD. Prove that BD bisects $\angle CDA$.

Ex. 12. By use of ruler and compasses construct a triangle whose sides are $1\frac{1}{4}$ in., $1\frac{3}{4}$ in., and $1\frac{1}{2}$ in. Through each vertex of the triangle draw a line parallel to the side opposite in the triangle. Produce the lines drawn through the vertices till they meet. How many triangles are now on the diagram? How many of these triangles are congruent? Prove a pair of these triangles congruent.

Ex. 13. Give a complete proof of § 105, by use of § 102.

Proposition XVI. Theorem

108. *The sum of the three angles of a triangle is equal to two right angles.*



Given the $\triangle ABC$.

To prove $\angle A + \angle B + \angle BCA = 2$ rt. $\measuredangle$.

Proof. 1. Produce AC through C to D.	1. Post. 2.
2. From C draw $CE \parallel AB$.	2. § 100.
3. $\angle s + \angle r + \angle BCA = 2$ rt. $\measuredangle$.	3. The sum of all the $\measuredangle$ about a point on the same side of a st. line passing through the point is 2 rt. $\measuredangle$. (§ 68.)
4. $\angle s = \angle A$.	4. If two $\parallel$ lines are cut by a transversal, any two corr. $\measuredangle$ are equal. (§ 102.)
5. Also $\angle r = \angle B$.	5. Why?
6. Substituting in 3 for $\angle s$ its equal, $\angle A$, and for $\angle r$ its equal, $\angle B$, $\angle A + \angle B + \angle BCA = 2$ rt. $\measuredangle$.	6. Ax. 9.
	Q.E.D.

109. Cor. 1. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

110. Cor. 2. *The sum of any two angles of a triangle is less than two right angles.*

111. Cor. 3. *If one angle of a triangle is a right angle or an obtuse angle, each of the other two angles of the triangle must be acute.*

112. Cor. 4. *In a right triangle, the sum of the two acute angles equals one right angle;*

Each angle of an equilateral triangle contains 60°.

113. Cor. 5. *If two angles of one triangle are respectively equal to two angles of another triangle, the third angle of the first triangle equals the third angle of the second.*

114. Cor. 6. *If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the triangles are equal.*

115. Cor. 7. *If two triangles have two angles and a side of one equal to two angles and a corresponding side of the other, the triangles are congruent.*

116. Cor. 8. *If two right triangles have the hypotenuse and an acute angle of one respectively equal to the hypotenuse and an acute angle of the other, the triangles are congruent.*

117. Cor. 9. *If two right triangles have a leg and an acute angle of one respectively equal to a leg and the corresponding acute angle of the other, the triangles are congruent.*

Ex. 1. If two angles of a triangle are 56° and 62°, find the remaining angle.

Ex. 2. If one acute angle of a right triangle is 36° 15′, find the other acute angle.

Ex. 3. How many degrees are there in each angle of an equiangular triangle?

Ex. 4. How many degrees are there in each acute angle of an isosceles right triangle?

Ex. 5. Is it possible to have a triangle whose angles are 45°, 62°, 72°?

Ex. 6. If one angle of a triangle is 42°, find the sum of the other two angles.

Ex. 7. If two angles of a triangle are 38° and 65°, find all the exterior angles of the triangle.

Ex. 8. If the vertex angle of an isosceles triangle is 38°, find each angle at the base.

Ex. 9. If an angle at the base of an isosceles triangle is 50°, find the vertex angle.

Ex. 10. An exterior angle at the base of an isosceles triangle is 102°. Find all the angles of the triangle.

Ex. 11. Construct an angle of 60°. Of 30°. Of 15°.

Ex. 12. Construct an angle of 120°. Of 75°.

Ex. 13. Construct an angle of 150°. Of 195°.

Ex. 14. Construct a triangle in which two of the sides are 1 inch and $1\frac{1}{2}$ inches, with an angle of 135° included between them.

Ex. 15. Construct a right angle and divide it into three equal parts.

Ex. 16. Construct an isosceles triangle in which the vertex angle is 120°.

Ex. 17. Construct an isosceles right triangle each of whose legs is $1\frac{1}{2}$ inches. How many degrees are in each of its angles?

Ex. 18. Construct a square whose side is $1\frac{1}{4}$ inches.

Ex. 19. Draw a line, AB, 2 inches long. Then, by aid of the protractor, construct a triangle in which $\angle A = 65°$ and $\angle B = 50°$. Also on another base $DE = \frac{3}{4}$ inch, construct a triangle DEF in which $\angle D = 65°$, and $\angle E = 50°$. Are each of the three angles of the triangle ABC equal to the corresponding angle in the triangle DEF? Are the triangles congruent?

Ex. 20. Draw any acute angle and in the shortest way construct its complement.

Ex. 21. Draw any acute angle and in the shortest way construct its supplement.

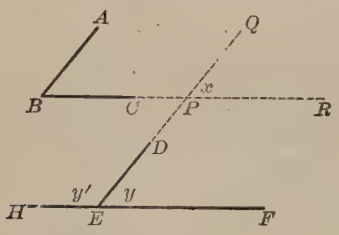
Ex. 22. Make up and solve an example similar to Ex. 5.

Ex. 23. To Ex. 14. To Ex. 19.

Ex. 24. Make up and solve two examples entirely your own concerning the numerical properties of the angles of a triangle.

Proposition XVII. Theorem

118. *If two angles have their sides respectively parallel, they are either equal or supplementary.*

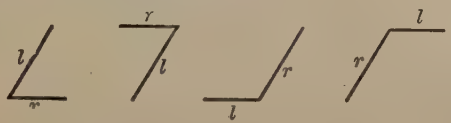


Given $AB \parallel ED$ and $BC \parallel HF$.

To prove $\angle B = \angle y$, and $\angle B$ is supplement of $\angle y'$.

Proof. 1. Produce the lines BC and ED to intersect in P.	1. Post. 2.
2. Then $\angle B = \angle x$.	2. § 102.
3. $\angle x = \angle y$.	3. Why?
4. $\therefore \angle B = \angle y$.	4. Why?
5. $\angle y$ is supplement of $\angle y'$.	5. § 32.
6. $\therefore \angle B$ is supplement of $\angle y'$.	6. Ax. 9.
	Q.E.D.

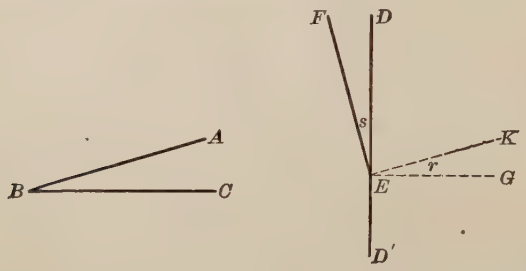
119. Note.—In the above diagram, if the eye of an observer is at B, BC will be the right side and BA the left side of $\angle ABC$.



Hence, in general, if the right and left sides of one angle are parallel, respectively, to the right and left sides of another angle, the angles are equal; but if the right and left sides of an angle are parallel, respectively, to the left and right sides of the other, the angles are supplementary.

Proposition XVIII. Theorem

120. *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*



Given $BA \perp EF$, and $BC \perp DD'$.

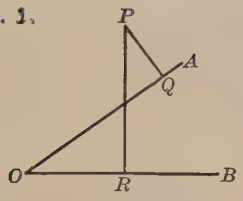
To prove $\angle B = \angle FED$, and $\measuredangle B$ and FED' supplementary.

Proof. 1. At E construct $EK \perp EF$ and in the same direction with BA; also construct $EG \perp DD'$ and in the same direction with BC.	1. At a given point of a given line, to erect a perpendicular to the line. (§ 89.)
2. Then $BA \parallel EK$, and $BC \parallel EG$.	2. Two lines in the same plane $\perp$ the same line are $\parallel$. (§ 97.)
3. $\therefore \angle B = \angle r$.	3. § 118.
4. But $\angle r$ is comp. of $\angle DEK$.	4. § 31.
5. Also $\angle s$ is comp. of $\angle DEK$.	5. Why?
6. $\therefore \angle r = \angle s$.	6. Why?
7. $\therefore \angle B = \angle s$, or $\angle FED$.	7. Ax. 1.
8. But $\angle FED'$ is sup. of $\angle s$.	8. § 32.
9. $\therefore \angle FED'$ is sup. of $\angle B$.	9. Ax. 9. **Q.E.D.**

121. Note.—The rule given in § 119 is true also if, in it, we change the word "parallel" to "perpendicular."

EXERCISES: GROUP 15

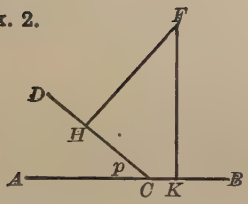
Ex. 1.



Given $\angle AOB$, $PQ \perp OA$, $PR \perp OB$.

Prove $\angle P = \angle O$.

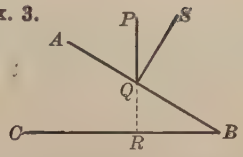
Ex. 2.



Given $FK \perp$ line ACB, $FH \perp$ line DHC.

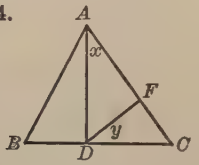
Prove $\angle F = \angle p$.

Ex. 3.



Given line $PQR \perp BC$, $SQ \perp AB$. What $\angle$ on the diagram $= \angle PQS$? Prove your statement.

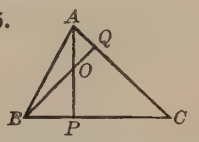
Ex. 4.



Given $\triangle ABC$, $AD \perp BC$, $DF \perp AC$.

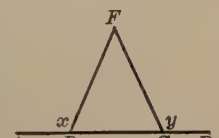
Prove $\angle x = \angle y$.

Ex. 5.



Given AP and BQ, altitudes of the $\triangle ABC$. Name the pairs of equal $\measuredangle$ on the diagram.

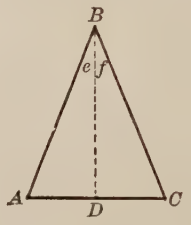
Ex. 6. Given $\triangle BFC$, $ABCD$ a straight line, $BF = FC$. Prove $\angle x = \angle y$. Also, make $AB = CD$, draw AF and FD, and prove $AF = FD$.



Ex. 7. By the aid of squared paper, construct the following designs:

Proposition XIX. Theorem

122. *If two angles of a triangle are equal, the sides opposite these angles are equal.*



Given $\triangle ABC$ in which $\angle A = \angle C$.

To prove $BA = BC$.

Proof. 1. Draw BD bisecting $\angle ABC$ and meeting AC in D.	1. To bisect a given angle (§ 88.)
2. Then in $\triangle ABD$ and DBC, $BD = BD$.	2. Why?
3. $\angle A = \angle C$.	3. Why?
4. $\angle e = \angle f$.	4. Why?
5. $\therefore \triangle ABD \cong \triangle DBC$.	5. § 115.
6. $\therefore BA = BC$.	6. Corr. sides of congruent $\triangle$ are $=$. Q.E.D.

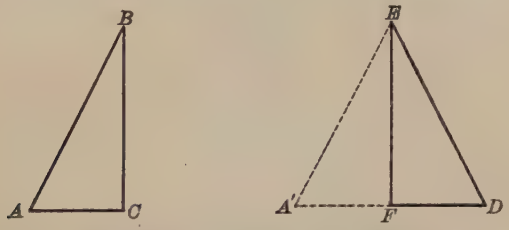
123. Cor. *An equiangular triangle is equilateral.*

124. The **converse of a proposition** is another proposition formed by interchanging the hypothesis and the conclusion of the original proposition. Thus § 122 is the converse of § 84.

The converse of a theorem is not always true. Thus the converse of § 118 is not true. State the converse of § 118. Give another theorem whose converse is not true.

Proposition XX. Theorem

125. *If two right triangles have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the triangles are congruent.*



Given two right $\triangle ABC$ and DEF having the hypotenuse AB = hypotenuse DE, and $BC = EF$.

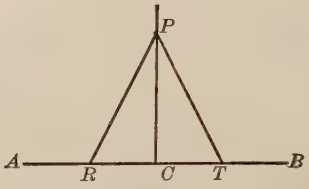
To prove $\triangle ABC \cong \triangle DEF$.

Proof. 1. Place the $\triangle ABC$ so that BC coincides with its equal, EF, and A falls on the opposite side of EF from D, at A'.	1. Geom. Ax. 2.
2. $\angle EFD$ is a rt. $\angle$, and $\angle EFA'$ is a rt. $\angle$.	2. Hyp.
3. $\therefore \angle A'FD$ is a st. $\angle$.	3. Ax. 2.
4. Then $A'F$ and FD will form a straight line, $A'FD$.	4. If two adj. $\measuredangle$ are together equal to a st. $\angle$, their ext. sides form a st. line. (§ 65.)
5. But $A'E = ED$.	5. Hyp.
6. $\therefore \angle A' = \angle D$.	6. § 84.
7. $\therefore \triangle A'EF$, or $\triangle ABC$, $\cong \triangle DEF$.	7. § 116.
	Q.E.D.

Ex. The perpendicular from the vertex of an isosceles triangle to the base divides the triangle into two congruent right triangles.

Proposition XXI. Theorem

126. *If two oblique straight lines drawn from a point to a straight line meet the line at equal distances from the foot of the perpendicular drawn from the point to the line, they are equal.*



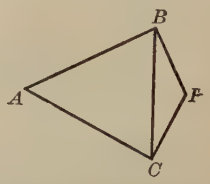
Given $PC \perp AB$, PR and PT oblique to AB, and $RC = CT$.

To prove $PR = PT$.

Proof. 1.	In the $\triangle RPC$ and CPT, $PC = PC$.	1. Why?
2.	$RC = CT$.	2. Why?
3.	$PC \perp RT$.	3. Why?
4.	$\therefore$ $\measuredangle PCR$ and PCT are right $\measuredangle$.	4. § 24.
5.	$\therefore$ $\angle PCR = \angle PCT$.	5. § 63.
6.	$\triangle RPC \cong \triangle CPT$.	6. Why?
7.	$PR = PT$.	7. Why?

Q.E.D

Ex. 1. **Given** the quadrilateral $ABPC$ with $AB = AC$, and $\measuredangle ABP$ and ACP right $\measuredangle$. **Prove** $BP = PC$.



Ex. 2. In the $\triangle PQR$, $PQ = QR$. The angles P and R are bisected by lines which meet at O. Prove $OP = OR$.

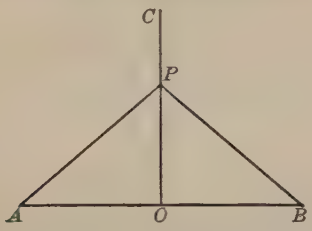
Ex. 3. Prove the converse of § 126; that is, on the figure of § 126, given $PC \perp AB$ and $PR = PT$, **prove** $RC = CT$.

Ex. 4. Construct the quadrilateral $ABCD$ in which $\angle A = 60°$, $AB = 2$ in., $BC = 1\frac{1}{2}$ in., $CD = 1\frac{1}{4}$ in., and $AD = 1\frac{1}{2}$ in.

Proposition XXII. Theorem

127. I. *Every point in the perpendicular bisector of a line is equally distant from the extremities of the line;* and

II. Conversely, *every point equidistant from the extremities of a line lies in the perpendicular bisector of the line.*



I. **Given** O the midpoint of the line AB, $OC \perp AB$, P any point in OC.

To prove $AP = PB$.

Proof. Let the student prove $\triangle APO \cong \triangle POB$, etc.

II. **Given** O the midpoint of AB, and $AP = PB$.

To prove $PO \perp AB$.

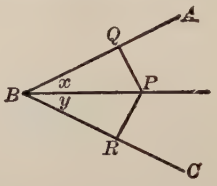
Proof. 1.	In the $\triangle APO$ and OPB, $PA = PB$.	1. Why?
2.	$PO = PO$.	2. Why?
3.	$AO = OB$.	3. Why?
4.	$\therefore \triangle APO \cong \triangle OPB$.	4. Why?
5.	$\therefore \angle AOP = \angle POB$.	5. Why?
6.	$\therefore \angle AOP$ is a rt. $\angle$.	6. Def. (§ 23.)
7.	$\therefore PO \perp AB$.	7. Def. (§ 24.)
		Q.E.D.

128. Cor. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

Proposition XXIII. Theorem

129. I. *Every point in the bisector of an angle is equidistant from the sides of the angle;* and

II. Conversely, *every point equidistant from the sides of an angle lies in the bisector of the angle.*



I. **Given** PB the bisector of the angle ABC, P any point in PB, $PQ \perp BA$, and $PR \perp BC$.

To prove $PQ = PR$.

Proof.	1. In the rt. $\triangle PBQ$ and PBR, $PB = PB$.	1. Why?
2.	$\angle x = \angle y$.	2. Why?
3.	$\therefore \triangle PBQ \cong \triangle PBR$.	3. Why?
4.	$\therefore PQ = PR$.	4. Why?

II. **Given** $\angle ABC$, $PQ \perp AB$, $PR \perp BC$, and $PQ = PR$.

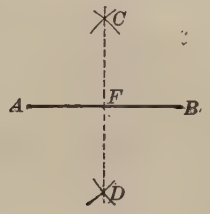
To prove that PB is the bisector of $\angle ABC$.

Proof.	1. In the rt. $\triangle PBQ$ and PBR, $PB = PB$.	1. Why?
2.	$PQ = PR$.	2. Why?
3.	$\therefore \triangle PBQ \cong \triangle PBR$.	3. Why?
4.	$\therefore \angle x = \angle y$.	4. Why?
Or BP is the bisector of $\angle ABC$.		Q.E.D

Ex. The point O is taken inside an equilateral triangle ABC so that $\angle OAB = \angle OBA$. Prove that OC bisects $\angle ACB$.

Proposition XXIV. Problem

130. *To bisect a given line-segment.*



Given the line AB.

To bisect AB.

Construction. 1. With A and B as centers and with equal radii greater than $\frac{1}{2}\,AB$, describe arcs intersecting in the points C and D.	1. Post. 3.
2. Draw CD intersecting AB in F. Then AB is bisected at F.	2. Post. 1.
Proof. 1. C is equidistant from A and B.	1. Constr.
2. D is equidistant from A and B.	2. Constr.
3. CD is $\perp$ bisector of AB.	3. § 128.
	Q.E.F.

Ex. 1. By use of ruler and compasses, find a point equidistant from three given points not in the same straight line.

Ex. 2. In the quadrilateral $PAQB$, $AP=PB$ and $AQ=QB$. Prove that PQ is the perpendicular bisector of AB.

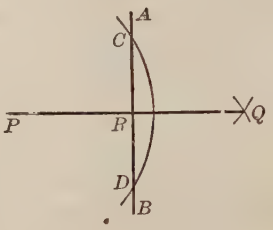
Ex. 3. In the triangle RST, $RS=ST$, and $\angle RST=48°$. RT is produced through T to Z, making $TZ=ST$, and SZ is drawn. Find all the angles of the figure.

Ex. 4. FGH is a triangle having $FG=HF$. GK is drawn $\perp FG$ and meeting FH produced at K, and HL is drawn $\perp FH$ and meeting FG produced at L. Prove $GK=HL$. How many isosceles triangles are on the figure? Prove this.

Ex. 5. In a given right triangle one leg is 12 and one acute angle is 45°. Find the other leg.

Proposition XXV. Problem

131. *To draw a perpendicular to a given line through a given external point.*



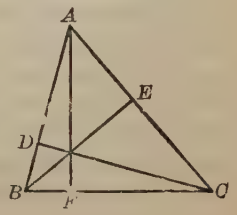
Given the line AB and the point P outside AB.

To construct a perpendicular from the point P to the line AB.

Construction. 1. With P as a center and with any convenient radius, describe an arc intersecting AB in two points, as C and D.	1. Post. 3.
2. With C and D as centers and with convenient equal radii greater than $\frac{1}{2}CD$, describe arcs intersecting at Q.	2. Post. 3.
3. Draw the line PQ meeting AB at R. Then PR is the $\perp$ required.	3. Post. 1.
Proof. 1. P is equidistant from C and D; also Q is equidistant from C and D.	1. Why?
2. PR is the $\perp$ bisector of CD, and $PR \perp AB$.	2. § 128.

Q.E.F.

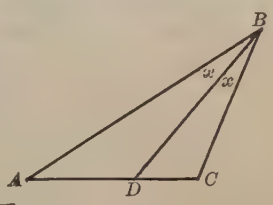
132. Altitudes of a triangle.—The altitude of a triangle has been defined in § 75 (p. 31). In any triangle, any side may be taken as the base. Hence the **altitudes** of a triangle are the three perpendiculars drawn one



from each vertex to the side opposite (or this side produced). Thus, in $\triangle ABC$, the three altitudes are AF, BE, and CD.

133. A **median** of a triangle is a line drawn from a vertex of the triangle to the midpoint of the opposite side. How many medians has a triangle? Draw a triangle and then construct its medians.

134. An **angle-bisector** of a triangle is a line which bisects an angle of the triangle. An angle-bisector is usually regarded as produced to meet that side of the triangle which is opposite the bisected angle. Thus BD is an angle-bisector of the triangle ABC. How many angle-bisectors has a triangle?



Ex. 1. Construct an equilateral triangle in which each side is half of a given line.

Ex. 2. Draw an obtuse triangle and construct its three medians.

Ex. 3. Draw an obtuse triangle and construct its three altitudes.

Ex. 4. Draw an isosceles triangle and construct its three angle-bisectors.

135. Inequality axioms.—To the list of GENERAL AXIOMS (p. 21), the following axioms, relating to unequal magnitudes, are to be added:

INEQUALITY AXIOM 1. *If equals are added to, or subtracted from, unequals, the results are unequal in the same order. If unequals are added to unequals in the same order, the results are unequal in that order.*

INEQUALITY AXIOM 2. *Doubles, or halves, of unequals are unequal in the same order.*

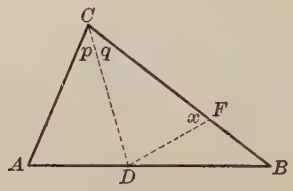
INEQUALITY AXIOM 3. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

INEQUALITY AXIOM 4. *If, of three quantities, the first is greater than the second, and the second is greater than the third, then the first is greater than the third.*

INEQUALITY AXIOM 5. *A given magnitude, when compared with another magnitude of the same kind, must be either equal to, less than, or greater than the other magnitude.*

PROPOSITION XXVI. THEOREM

136. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*



Given $\triangle ABC$ with side $BC >$ side AC.

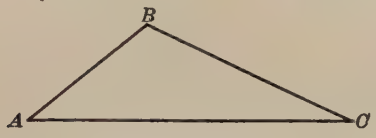
To prove $\angle A > \angle B$.

Proof. 1. Draw CD bisecting $\angle ACB$ and meeting AB in D.	1. § 88.
2. On CB lay off $CF = AC$. Draw DF.	2. Posts. 2, 1.
3. In $\triangle ADC$ and DFC, $AC = CF$.	3. Why?
4. Also $CD = CD$.	4. Why?
5. $\angle p = \angle q$.	5. Why?
6. $\therefore \triangle ADC \cong \triangle DFC$.	6. Why?
7. Hence $\angle A = \angle x$.	7. Why?
8. But $\angle x$, being an ext. $\angle$ of $\triangle DFB$, is greater than $\angle B$.	8. § 91.
9. $\therefore \angle A > \angle B$.	9. Ax. 9.
	Q.E.D.

Ex. Which is the longest side of a right triangle? Of an obtuse triangle?

Proposition XXVII. Theorem

137. *If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.*



Given $\angle A$ greater than $\angle C$ in the $\triangle ABC$.

To prove $BC > AB$.

Proof. 1. Either $BC = AB$, or $BC < AB$, or $BC > AB$.	1. Ineq. Ax. 5. (§ 135.)
2. If $BC = AB$,	2. § 84.
then $\angle A = \angle C$.	
3. This is impossible.	3. For $\angle A > \angle C$, by hyp.
4. Also, if $BC < AB$,	4. § 136.
then $\angle A < \angle C$.	
5. But this is also impossible.	5. Reason 3.
6. $\therefore BC > AB$.	6. Since BC neither $= AB$, nor $< AB$. **Q.E.D.**

Ex. 1. State the inequality axiom which is illustrated in the following statement: If $7 > 5$, then $14 > 10$.

Ex. 2. Give a numerical illustration of each of the different statements in Ineq. Ax. 1.

Ex. 3. Give numerical illustrations of Ineq. Ax. 2. Ineq. Ax. 3.

Ex. 4. Give numerical illustrations of Ineq. Ax. 4.

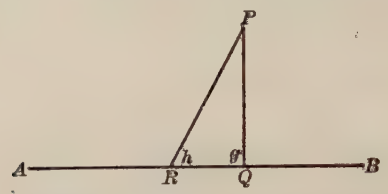
Ex. 5. The theorem of § 137 is the converse of what theorem?

Ex. 6. In the quadrilateral $PQRS$, PQ is the longest side and RS is the shortest. Prove that $\angle PSR$ is greater than $\angle PQR$.

[Sug. Draw QS.]

Proposition XXVIII. Theorem

138. *The perpendicular is the shortest line that can be drawn from a given point to a given straight line.*



Given the point P and the line AB. $PQ \perp AB$, and PR any other line from P to AB.

To prove $PQ < PR$.

Proof. 1. $\angle g$ is a rt. $\angle$.	1. Hyp.
2. $\angle h$ is an acute $\angle$.	2. If one angle of a $\triangle$ is a rt. $\angle$ or an obtuse $\angle$, each of the other two angles of the $\triangle$ must be acute. (§ 111.)
3. $\angle h < \angle g$.	3. An acute $\angle$ is an angle less than a rt. $\angle$. (§ 25.)
4. $\therefore PQ < PR$.	4. § 137. Q.E.D.

139. Cor. (Converse of Prop. XXVIII.) *If a line is the shortest line that can be drawn from a given point to a given line, it is the perpendicular from the point to the given line.*

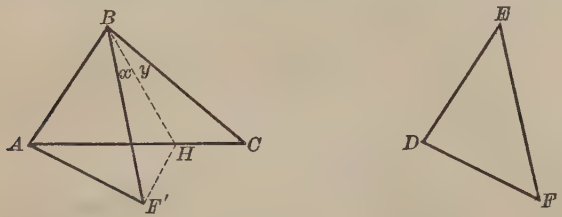
Thus, given PQ the shortest line from P to AB, then $PQ \perp AB$.

Ex. Give a formal proof of § 139.

[Sug. Use the indirect method. Thus, if PQ is not $\perp AB$, draw PT, a $\perp$ from P to AB. Then, in $\triangle PQT$, show that PQ would be greater than PT, etc.]

Proposition XXIX. Theorem

140. *If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given the $\triangle ABC$ and DEF in which $AB=DE$, $BC=EF$, and $\angle ABC$ is greater than $\angle E$.

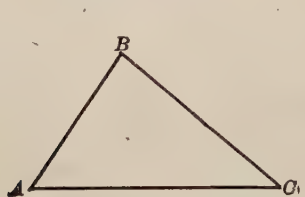
To prove $AC>DF$.

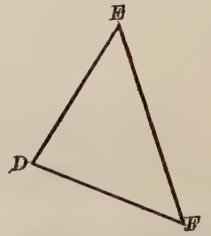
Proof. 1. Place the $\triangle DEF$ so that DE coincides with its equal, AB, and F falls on the same side of AB as C, at F'.	1. Geom. Ax. 2.
2. Construct the line BH bisecting $\angle F'BC$ and meeting the line AC at H. Draw $F'H$.	2. § 88, Post. 1.
3. Then, in the $\triangle F'BH$ and BHC, $F'B=BC$.	3. Hyp.
4. $BH=BH$.	4. Why?
5. $\angle x=\angle y$.	5. Why?
6. $\therefore \triangle F'BH\cong\triangle BHC$.	6. Why?
7. $\therefore F'H=CH$.	7. Why?
8. But $AH+HF'>AF'$.	8. Why?
9. Substituting for HF' its equal, HC, $AH+HC$, or AC, $>AF'$.	9. Why?
10. $\therefore AC>DF$.	10. Why?
	Q.E.D.

Ex. In the quadrilateral $ABCD$, $AB=AC=AD$, and $\angle BAC=\angle CAD=50°$. Find $\angle BCD$.

Proposition XXX. Theorem

141. *If two sides of one triangle are equal respectively to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*





Given the $\triangle ABC$ and DEF having $AB=DE$, $BC=EF$, but $AC>DF$.

To prove $\angle B>\angle E$.

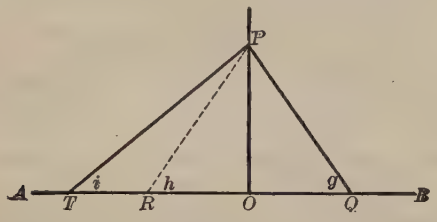
Proof. 1. Either $\angle B=\angle E$, or $\angle B<\angle E$, or $\angle B>\angle E$.	1. Ineq. Ax. 5. (§ 135.)
2. If $\angle B=\angle E$, then $\triangle ABC\cong\triangle DEF$, and $AC=DF$.	2. § 81. Corr. sides of congruent $\triangle$ are $=$.
3. But this is impossible.	3. For $AC>DF$, by hyp.
4. Also, if $\angle B<\angle E$, $AC<DF$.	4. § 140.
5. But this also is impossible.	5. Reason 3.
6. $\therefore \angle B>\angle E$.	6. For $\angle B$ neither $=\angle E$, nor is less than $\angle E$. Q.E.D.

Ex. 1. How many points in a plane are necessary to determine two parallel lines? Three parallel lines?

Ex. 2. Are two triangles necessarily congruent if three angles of one equal the corresponding three angles of the other? Illustrate by drawing a figure.

Proposition XXXI. Theorem

142. *If two oblique straight lines drawn from a point to a straight line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, the more remote is the greater.*



Given $PO \perp AB$, and T and Q points on AB such that $OT > OQ$.

To prove $PT > PQ$.

Proof. 1. On OA mark off $OR = OQ$.	1. Post. 2.
2. Draw PR.	2. Post. 1.
3. Then $PR = PQ$.	3. § 126.
4. $\angle h = \angle g$.	4. § 84.
5. But $\angle h > \angle i$.	5. § 91.
6. Substituting $\angle g$ for its equal, $\angle h$, $\angle g > \angle i$.	6. Ax. 9.
7. Then, in $\triangle TPQ$, $PT > PQ$.	7. § 137.
	Q.E.D.

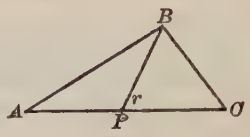
143. Cor. *Of two unequal lines drawn to a given line from a point in a perpendicular to that line, the greater line cuts off the greater segment from the foot of the perpendicular.*

Thus, if $PT > PQ$ (Fig. of Prop. XXXI), OT cannot $= OQ$ (§ 126); nor is $OT < OQ$ (§ 142). $\therefore OT > OQ$.

Hence, also, *from a given point only two equal straight lines can be drawn to a given line.*

EXERCISES: GROUP 16

Ex. 1. Given $\triangle ABC$, $AP=PC$, $\angle r$ acute. Prove $AB>BC$.

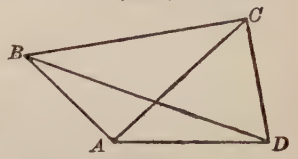


(Ex. 1)

Ex. 2. Draw any quadrilateral $ABCD$; let F be any point in the side BC. Prove the sum of the sides of $ABCD$ > the sum of the sides of AFD.

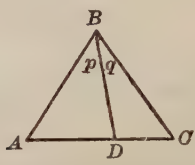
Ex. 3. Draw a triangle ABC; let D be any point in AB and F any point in BC. Prove $AB+BC>AD+DF+FC$.

Ex. 4. Given a quadrilateral $ABCD$, $AB=CD$, $\angle BAD>\angle ADC$. Prove $BD>AC$.

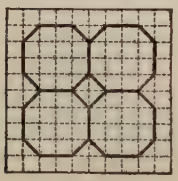


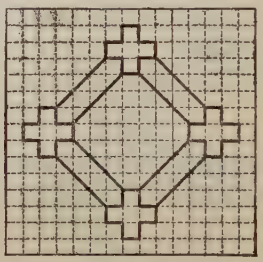
(Ex. 4)

Ex. 5. Given $\triangle ABC$, $AB=BC$, $AD>DC$. Prove $\angle p>\angle q$.



Ex. 6. By the aid of squared paper, construct the following designs:





QUADRILATERALS

144. A **quadrilateral** is a polygon of four sides.

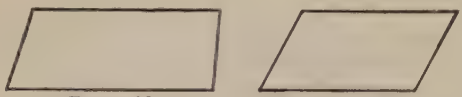
The **sides** of a quadrilateral are the bounding lines. The **angles** are the angles made by the consecutive bounding lines. The **vertices** are the vertices of the angles of the quadrilateral.

The **perimeter** of a quadrilateral is the sum of the sides.

145. A **diagonal** of a quadrilateral is a straight line joining two of its vertices that are not adjacent.

146. A **trapezoid** is a quadrilateral which has two, and only two, of its sides parallel.

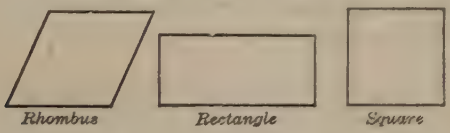
147. A **parallelogram** is a quadrilateral whose opposite sides are parallel.



148. A **rhombus** is a parallelogram whose sides are equal.

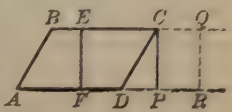
149. A **rectangle** is a parallelogram whose angles are right angles.

A **square** is a rectangle whose sides are equal.



150. The **base of a parallelogram** is the side upon which the parallelogram is supposed to stand; as AD. The opposite side is called the upper base (BC).

The **altitude** of a parallelogram is the perpendicular distance between the bases, or bases produced. Thus the altitude of the parallelogram $ABCD$ is EF, or CP, or QR.



151. The **bases of a trapezoid** are its two parallel sides. The **legs** of a trapezoid are the sides which are not parallel.

The **altitude** of a trapezoid is the perpendicular distance between the bases. The **median** of a trapezoid is the line joining the midpoints of the legs.

152. An **isosceles trapezoid** is a trapezoid whose legs are equal.

Ex. 1. Draw a quadrilateral with three acute angles and one obtuse angle.

Ex. 2. Construct an isosceles trapezoid whose bases are 3 inches and 2 inches, and whose altitude is $1\frac{1}{2}$ inches.

Ex. 3. Find the perimeter of a square foot in inches.

Ex. 4. Into how many triangles does a diagonal of a rhombus divide the rhombus? What kind of triangles are these?

Ex. 5. Construct a parallelogram $ABCD$. Let P be the midpoint of the side BC and Q the midpoint of AD. Draw AC, BQ, and DP. Denote the angles of the figure by small letters like a, b, c, etc. On the figure thus constructed point out 8 pairs of alternate interior angles. Also 6 pairs of corresponding angles. Also 4 pairs of interior angles on the same side of a transversal.

Ex. 6. Construct two parallelograms whose corresponding sides are equal, but whose corresponding angles are unequal.

Ex. 7. Prove that the sum of the angles of any quadrilateral is two straight angles.

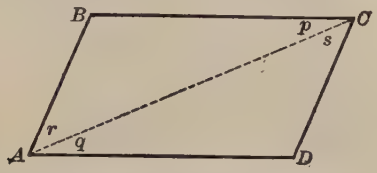
Ex. 8. If one angle of a quadrilateral is 60°, and the other three angles are equal, find these angles.

Ex. 9. In Ex. 10 on p. 28, prove that the error in the outside angle of the carpenter's square, if there is any, equals one half the angle (x) between the outside lines of the square as shown in the diagram. (Denote the error by e and show that $e+e=x$.)

This principle is important because it is essentially the method used in correcting the axis of a telescope, and hence in correcting various surveying and astronomical instruments of which the telescope is a part.

Proposition XXXII. Theorem

153. *The opposite sides of a parallelogram are equal, and the opposite angles are equal.*



Given the parallelogram $ABCD$.

To prove $AD=BC$, $AB=DC$, $\angle B=\angle D$, and $\angle BAD=\angle BCD$.

Proof. 1. Draw the diagonal AC.	1. Post. 1.
2. Then, in the $\triangle ABC$ and ADC, $AC=AC$.	2. Why?
3. Also $\angle p=\angle q$.	3. § 101.
4. $\angle r=\angle s$.	4. Why?
5. $\therefore \triangle ABC\cong\triangle ACD$.	5. Why?
6. $\therefore AD=BC$, $AB=DC$, $\angle B=\angle D$.	6. Corr. parts of congruent $\triangle$ are $=$.
7. In like manner, by drawing the diagonal BD, it may be proved that $\angle BAD=\angle BCD$.	7. Reasons 1–6.
	Q.E.D.

154. Cor. 1. *A diagonal divides a parallelogram into two congruent triangles.*

155. Cor. 2. *Parallel lines comprehended between parallel lines are equal.*

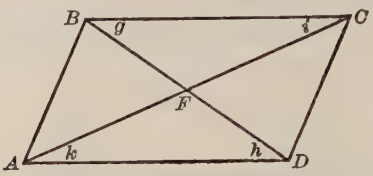
156. Cor. 3. *Two parallel lines are everywhere equidistant.*

Ex. 1. In the above figure, prove $\angle BAD=\angle BCD$ by use of Ax. 2.

Ex. 2. In the above diagram, produce BC and DC through C and prove $\angle BAD=\angle BCD$ by use of §§ 118 and 69.

Proposition XXXIII. Theorem

157. *The diagonals of a parallelogram bisect each other.*



Given the diagonals AC and BD of the $\square ABCD$, intersecting at F.

To prove $AF = FC$, and $BF = FD$.

Proof. Let the student supply the proof.

[Sug. In the $\triangle BFC$ and AFD, what sides are equal, and why? What $\angle$ are equal, and why? etc.]

Ex. 1. How many pairs of congruent triangles are there in the above figure?

Ex. 2. If two angles of a triangle are $p°$ and $q°$, find the third angle.

Ex. 3. If two angles of a triangle are $x°$ and $90°+x°$, find the third angle in terms of x.

Ex. 4. Draw the figure and letter it for the following theorem: "The diagonals of a rectangle are equal." Then, in terms of the figure drawn, state the hypothesis and conclusion (but do not give proof).

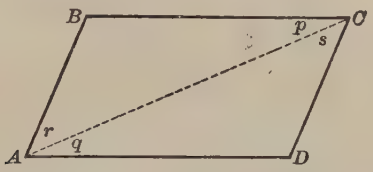
Ex. 5. By use of the ruler and compasses construct an angle of 105°. (Observe that $105° = 60° + 45°$.)

Ex. 6. Construct a parallelogram two of whose sides are 2 inches and 1 inch and one of whose angles is 60°.

Ex. 7. Construct two triangles which have two sides of one triangle equal to two sides of the other, and the angles opposite one pair of corresponding sides equal, but having the equal parts so arranged that the triangles themselves are not congruent.

Proposition XXXIV. Theorem

158. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ in which $AB=CD$ and $BC=AD$.

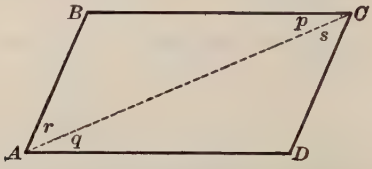
To prove $ABCD$ a $\square$.

Proof. 1. Draw the diagonal AC.	1. Post. 1.
2. In $\triangle ABC$ and ADC, $AC=AC$.	2. Why?
3. $BC=AD$.	3. Why?
4. $AB=CD$.	4. Why?
5. $\therefore \triangle ABC \cong \triangle ADC$.	5. Why?
6. $\therefore \angle r=\angle s$.	6. Why?
7. $\therefore AB \parallel CD$.	7. If two lines are cut by a transversal so that a pair of alt. int. $\measuredangle$ are equal, the two lines are parallel. (§ 93.)
8. Also $\angle p=\angle q$.	8. Why?
9. $\therefore BC \parallel AD$.	9. Why?
10. $\therefore ABCD$ is a $\square$.	10. § 147. **Q.E.D.**

Ex. State in order the theorems and other geometric principles used in proving that if two triangles have the three sides of one respectively equal to the three sides of the other, the triangles are congruent.

Proposition XXXV. Theorem

159. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ in which $BC =$ and $\| AD$.

To prove $ABCD$ a $\square$.

Proof. 1.	Draw the diagonal AC.	1. Post. 1.
2.	In the $\triangle ABC$ and ADC, $AC = AC$.	2. Why?
3.	$BC = AD$.	3. Why?
4.	$\angle p = \angle q$.	4. Why?
5.	$\therefore \triangle ABC \cong \triangle ADC$.	5. Why?
6.	$\therefore \angle r = \angle s$.	6. Why?
7.	$\therefore AB \| CD$.	7. Why?
8.	$\therefore ABCD$ is a $\square$.	8. § 147.

Q.E.D.

Ex. 1. Show that in a parallelogram each pair of adjacent angles is supplementary.

Ex. 2. One angle of a parallelogram is 43°; find the other angles.

Ex. 3. If one angle of a parallelogram is three times another angle, find all the angles of the parallelogram.

Ex. 4. If one angle of a parallelogram is $a°$, find the other angles.

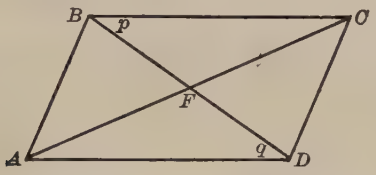
Ex. 5. If, in the triangle ABC, $\angle A = 60°$ and $\angle B = 70°$, which is the longest side in the triangle? Which is the shortest side?

Ex. 6. In the diagram on p. 90 point out 6 pairs of corresponding angles. Also 6 pairs of alternate interior angles.

Ex. 7. Make up and solve an example similar to Ex. 5.

Proposition XXXVI. Theorem

160. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ in which the diagonals AC and BD bisect each other at F.

To prove $ABCD$ a $\square$.

Proof. 1. In the $\triangle BFC$ and AFD, $BF = FD$.	1. Why?
2. $CF = AF$.	2. Why?
3. $\angle BFC = \angle AFD$.	3. Why?
4. $\therefore \triangle BFC \cong \triangle AFD$.	4. Why?
5. $\angle p = \angle q$.	5. Why?
6. $\therefore BC \parallel AD$.	6. Why?
7. In like manner, by use of the $\triangle ABF$ and FCD, it may be proved that $AB \parallel CD$.	7. Reasons 1–6.
8. $\therefore ABCD$ is a $\square$.	8. § 147. **Q.E.D.**

Ex. 1. Prop. XXXVI is the converse of what theorem?

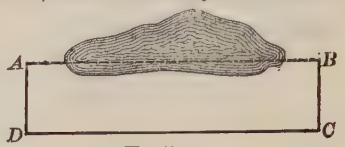
Ex. 2. Is the converse of a proposition always true? Give two propositions which illustrate your statement.

Ex. 3. The efficiency value of § 158 is that it enables us to ascertain that the opposite sides of a given quadrilateral are parallel by measuring the length of the sides of the quadrilateral. State the efficiency value of § 160.

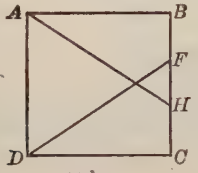
EXERCISES: GROUP 17

Ex. 1. To determine the distance AB, construct AD and $BC \perp AE$, making $AD = BC$. Prove $ABCD$ a parallelogram.

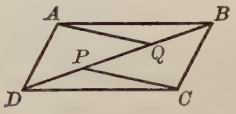
If we measure DC and find it to be 312 yd., how long is AB? Why?



(Ex. 1)



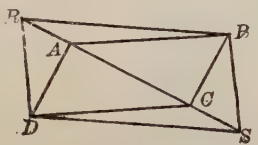
(Ex. 2)



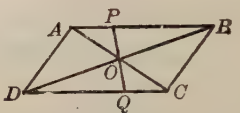
(Ex. 3)

Ex. 2. Given $ABCD$ a square, and $BF = HC$. Prove $AH = DF$.

Ex. 3. Given $ABCD$ a parallelogram; on the diagonal DB, $DP = BQ$. Prove $AQ = CP$.



(Ex. 4)



(Ex. 5)

Ex. 4. Given $ABCD$ a parallelogram. The diagonal AC is produced so that $AR = CS$. Prove $RBSD$ a parallelogram.

Ex. 5. Given $ABCD$ a $\square$ with its diagonals intersecting at O; PQ any line through O terminated by the sides of the $\square$. Prove $OP = OQ$.

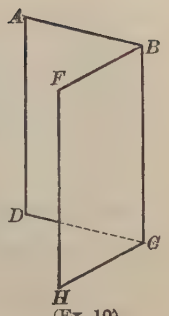
Ex. 6. Prove that the diagonals of a rhombus are perpendicular to each other, by the use of § 128.

Ex. 7. $ABCD$ is a rectangle and its diagonals intersect at O. Prove $AO = OB$.

Ex. 8. $ABCD$ is a parallelogram with AB produced through B to P and CD through D to Q. $BP = DQ$. Prove $\angle QAD = \angle BCP$.

Ex. 9. In the triangle AHD, B is the midpoint of AH and K of HD. BK is produced to C, and $CD \parallel AH$. Find two lines on the figure each equal to CD.

Ex. 10. Given $ABCD$ and $FBCH$, parallelograms with the side BC in common. **Prove** the line AD = and $\parallel FH$.



(Ex. 10)

Ex. 11. Prove that the sum of the four sides of a quadrilateral is greater than the sum of the diagonals.

Ex. 12. Upon a given straight line as a side, construct a square.

Ex. 13. Construct a rectangle whose base is 2 in. and whose altitude is 1 in.

Ex. 14. Upon a given line as a diagonal, construct a square.

Ex. 15. Construct a rhombus, given the two diagonals.

POLYGONS.

161. A polygon, as already defined, is a closed broken line; as $ABCDE$ or $FGHIJK$.

The **sides** of a polygon are its bounding lines. The **perimeter** of a polygon is the sum of its sides. The **angles** of a polygon are the angles formed by its consecutive sides. The **vertices** of a polygon are the vertices of its angles.

A **diagonal** of a polygon is a straight line joining two vertices which are not adjacent; as BD in Fig. 1.

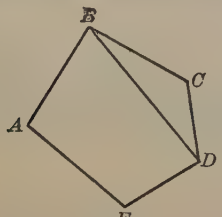


FIG. 1

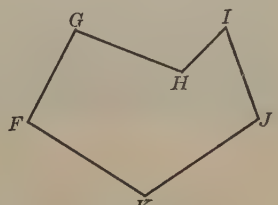


FIG. 2

162. An **equilateral polygon** is a polygon in which all the sides are equal.

163. An **equiangular polygon** is a polygon in which all the angles are equal.

What four-sided polygon is equilateral but not equiangular? What four-sided polygon is both equilateral and equiangular?

164. Two mutually equiangular polygons are polygons whose corresponding angles are equal; as Figs. 3 and 4.

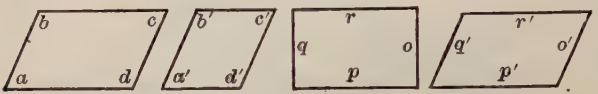


FIG. 3 FIG. 4 FIG. 5 FIG. 6

Two mutually equilateral polygons are polygons whose corresponding sides are equal; as Figs. 5 and 6.

From Figs. 3 and 4 it is seen that two polygons may be mutually equiangular without being mutually equilateral.

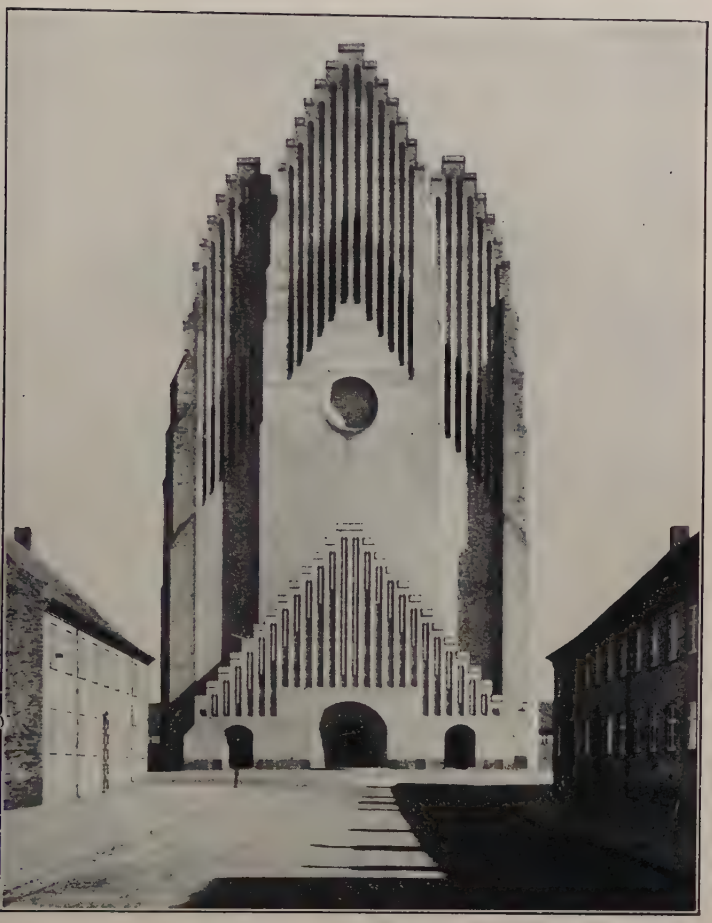
What similar truth may be inferred from Figs. 5 and 6?

Polygons that are mutually equiangular and equilateral may be made to coincide and are therefore congruent.

165. Names of polygons.—Some polygons are used so frequently that special names have been given to them. A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral**; one of five sides, a **pentagon**; of six sides, a **hexagon**; of seven sides, a **heptagon**; of eight sides, an **octagon**; of ten sides, a **decagon**; of twelve sides, a **dodecagon**; of fifteen sides, a **pentadecagon**; of n sides, an **n-gon.**

Ex. 1. How does the number of vertices of a polygon compare with the number of sides?

Ex. 2. Can two triangles be mutually equilateral without being mutually equiangular? What polygons can?



GEOMETRY IN ARCHITECTURE

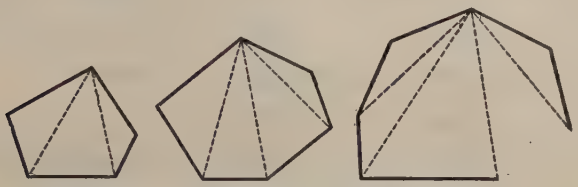
The architecture of this church in Copenhagen suggests a vast organ. What geometric figures are used to give this effect? (*Ewing Galloway*)

OUR MODERN PYRAMIDS

This huge hotel in New York is all rectangles. The long parallel lines of the windows seem to increase the height, while the symmetrical pyramiding of the upper stories gives grace as well as sunlight. (*Ewing Galloway*)

Proposition XXXVII. Theorem

166. *The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.*



Given a convex polygon of n sides.

To prove the sum of the angles of the polygon $=2(n-2)$ right angles.

Proof. 1. Separate the polygon into ▲ by diagonals drawn from some one vertex.	1. Post. 1.
2. There will be $(n-2)$ of these ▲.	2. Each of the n sides of the polygon will be the base of a △, except the two sides adjacent to the chosen vertex.
3. The sum of the ∠ of each △ is 2 rt. ∠.	3. The sum of the 3 ∠ of a △ $=2$ rt. ∠. (§ 108.)
4. The sum of the ∠ of $(n-2)$ ▲ $=2(n-2)$ rt. ∠.	4. Ax. 4.
5. But the sum of the ∠ of the polygon = the sum of the ∠ of the ▲.	5. The whole is equal to the sum of its parts. (Ax. 7.)
6. ∴ the sum of the ∠ of the polygon $=2(n-2)$ rt. ∠.	6. Ax. 1. Q.E.D.

167. Cor. *In an equiangular polygon of n sides, each angle equals* $\frac{2(n-2)\text{rt. } \angle}{n}$.

EXERCISES: GROUP 18

How many right angles are there in the sum of the angles of

Ex. 1. A quadrilateral?

Ex. 2. A hexagon?

Ex. 3. A sixteen-sided polygon?

Ex. 4. A polygon with 20 sides?

How many degrees are there in the sum of the angles of

Ex. 5. A pentagon?

Ex. 6. An octagon?

Ex. 7. A decagon?

Ex. 8. A polygon of 18 sides?

State the number of degrees in each angle of an equiangular

Ex. 9. Hexagon.

Ex. 10. Heptagon.

Ex. 11. Decagon.

Ex. 12. Polygon of 16 sides.

Ex. 13. If two angles of a quadrilateral are 73° and 106°, how many degrees are there in the sum of the other two angles of the quadrilateral?

Ex. 14. If three of the angles of a quadrilateral are 82°, 97°, and 63°, find the other angle of the quadrilateral without measuring it.

Ex. 15. Make up and solve a problem similar to Ex. 14 concerning a pentagon.

Ex. 16. Prove that if two of the angles of a quadrilateral are supplementary, the other two angles must be supplementary also.

Ex. 17. If one angle of a parallelogram is a right angle, prove that the figure is a rectangle.

Ex. 18. If one of the angles of an equiangular polygon contains 108°, how many sides has the polygon?

Ex. 19. Would a quadrilateral constructed of rods hinged at the ends (*i.e.*, at the vertices of the quadrilateral) be rigid? Would a triangle so constructed be rigid? Would a pentagon?

Ex. 20. In the quadrilateral $ABCD$, $AB=BC$, and DB bisects $\angle ABC$. Prove that $\triangle ADC$ is isosceles.

Ex. 21. AB and DF are lines intersecting at O. $AO=OB$ and $AD \parallel FB$. Prove $DO=OF$.

Ex. 22. R is a point inside the triangle ABC such that $RB = RC$ and $\angle BRA = \angle CRA$. Prove that AR bisects $\angle BAC$ and that $\triangle ABC$ is isosceles.

Ex. 23. $ABCD$ is a square with its diagonals intersecting at the point P. Prove the triangle APD isosceles.

Ex. 24. In the triangle ABC, $AB = BC$ and AB is produced through B to H and CB is produced through B to D. DH is parallel to AC. Prove $AD = HC$.

Ex. 25. By use of ruler and compasses construct the quadrilateral $ABCD$ so that $\angle BAD = 60°$, $AB = AD = 2\frac{1}{2}$ in., and $BC = CD = 1\frac{1}{2}$ in. Draw AC and BD. How many isosceles triangles are on the figure? How many pairs of congruent triangles? Prove two pairs of these triangles congruent.

Ex. 26. In the triangle ABC, D is a point on the side AB, and F is a point on AC, such that DF is parallel to BC, and $AD = AF$. Prove $AB = AC$.

Ex. 27. $ABCD$ is a rectangle. P is a point on the side AB and Q a point on CD, such that $BP = CQ$. Prove $\angle APD = \angle AQD$. How many other pairs of equal acute angles are on the figure? How many pairs of equal lines are on the figure? If AQ and PD intersect at O, prove $AO = OD$.

Ex. 28. $ABCD$ is a parallelogram. K is the midpoint of the side AB and L is the midpoint of the side CD. If the lines KD, KC, LA, and LB are drawn, how many parallelograms are on the figure?

Ex. 29. $ABCD$ is a parallelogram with AB produced through B to P and CD produced through D to Q. $QD = BP$ and PQ intersects AC at O. Prove $PO = OQ$.

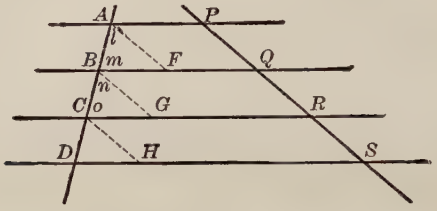
Ex. 30. By use of ruler and compasses construct a quadrilateral $ABCD$ in which the angle $BAD = 120°$, $AB = 2$ in., $BC = 1\frac{3}{4}$ in., $AC = 2\frac{1}{2}$ in., and $AD = 1\frac{1}{2}$ in.

Ex. 31. Prove that if a perpendicular is erected at any point in the base of an isosceles triangle, except the midpoint, and meets one side and another side produced, another isosceles triangle is formed, external to the given triangle.

Ex. 32. By use of ruler and compasses construct a parallelogram $ABCD$ in which $\angle BAD = 135°$, $AB = 1$ in., and $AD = 1\frac{3}{4}$ in.

Proposition XXXVIII. Theorem

168. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given AP, BQ, CR, and DS parallel lines intercepting the equal segments AB, BC, and CD on the transversal AD and the segments PQ, QR, RS on the transversal PS.

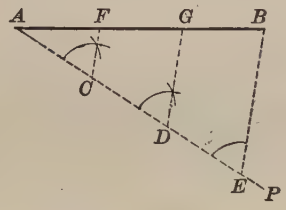
To prove $PQ=QR=RS$.

Proof. 1. Through A, B, C, construct AF, BG, CH, $\parallel PS$ and meeting BQ, CR, DS, in the points F, G, H, respectively.	1. § 100.
2. Then the lines AF and BG are $\parallel$.	2. § 107.
3. Then in the $\triangle ABF$ and BCG, $AB=BC$.	3. Why?
4. Also $\angle l=\angle n$, and $\angle m=\angle o$.	4. § 102.
5. $\therefore \triangle ABF\cong\triangle BCG$.	5. Why?
6. $\therefore AF=BG$.	6. Why?
7. But $AF=PQ$, and $BG=QR$.	7. § 155.
8. $\therefore PQ=QR$.	8. Ax. 1.
9. In like manner, it may be proved that $QR=RS$.	9. Reasons 2–8. Q.E.D.

169. Cor. *If a line is parallel to one side of a triangle and bisects a second side, it bisects the third side.*

Proposition XXXIX. Problem

170. *To divide a given line-segment into a given number of equal parts.*



Given the line AB.

To divide AB into a given number of equal parts (as three equal parts).

Construction. 1. Take a point P. Draw the line AP making a small angle with AB.	1. Post. 1.
2. Take AC (any line of convenient length) and apply it to AP a number of times equal to the number of parts into which AB is to be divided.	2. Post. 2.
3. From E, the end of the measure when last applied to AP, draw EB.	3. Post. 1.
4. Through the other points of division on AP (C and D), draw lines $\parallel EB$ and meeting AB at F and G.	4. § 100.
Then AB is divided into the required number of parts at F and G.	
Proof. 1. $AC=CD=DE$.	1. Constr.
2. $\therefore AF=FG=GB$.	2. § 168.
	Q.E.F.

Ex. 1. Draw a line 2 inches long and divide it into 3 equal parts.

Ex. 2. Draw a line 3″ long and divide it into 5 equal parts.

Ex. 3. Draw a line and then determine $\frac{2}{3}$ of the line.

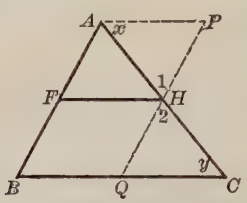
Ex. 4. Draw a line and determine $\frac{4}{5}$ of it.

Ex. 5. Draw a line and then construct a triangle whose sides shall be $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ of the given line.

Ex. 6. Given its perimeter, construct an equilateral triangle.

Proposition XL. Theorem

171. *The line joining the midpoints of two sides of a triangle is parallel to the third side, and equals one half the third side.*



Given $\triangle ABC$ with F and H the midpoints of AB and AC, respectively.

To prove $FH \parallel BC$, and $FH = \frac{1}{2}BC$.

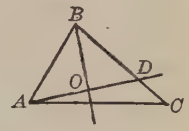
Proof. 1. Through A draw $AP \parallel BC$, and through H draw $PQ \parallel AB$ and meeting AP in P, and BC in Q.	1. § 100.
2. In $\triangle AHP$ and QHC, $AH = HC$.	2. Why?
3. Also $\angle 1 = \angle 2$,	3. Why?
and $\angle x = \angle y$.	
4. $\therefore \triangle APH \cong \triangle QHC$.	4. Why?
5. $\therefore HQ = HP$,	5. Why?
or $HQ = \frac{1}{2}PQ$.	
6. $PQ = AB$.	6. § 155.
7. $\therefore HQ = FB$.	7. Ax. 5.
8. $\therefore FHQB$ is a $\square$, and $FH \parallel BQ$.	8. §§ 159, 147.
9. $\therefore FH = BQ = AP = QC = \frac{1}{2}BC$.	9. Why?
	Q.E.D.

172. Proving triangles congruent.—Let the student form a list of the conditions that make two triangles congruent. (See §§ 81, 83, 87, 115, 116, 117, 125.)

EXERCISES: GROUP 19

Congruent Triangles

Ex. 1. Given ABC any triangle, BO the bisector of $\angle ABC$, and $AD \perp BO$. Prove $\triangle ABO \cong \triangle BOD$.

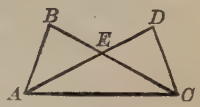


Ex. 2. If, upon the sides of an angle, equal segments are laid off from the vertex, and lines are drawn from the ends of these segments to any point in the bisector of the angle, prove that the triangles formed are congruent.

Ex. 3. If two sides of a triangle are produced, each its own length, through the vertex in which they meet, and the extremities of the produced parts are joined, prove that a new triangle is formed which is congruent with the original triangle.

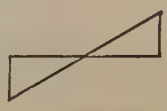
Ex. 4. Given $AB = DC$, and $BC = DA$. Prove $\triangle BAC \cong \triangle DAC$.

What other pair of congruent triangles is there in the figure?



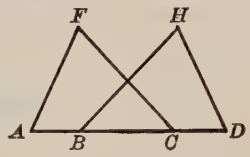
Ex. 5. If, at any point in the bisector of an angle, a $\perp$ is erected and produced to meet the sides of the angle, how many triangles are formed? Which of these triangles are congruent? Prove your statement.

Ex. 6. If, through the midpoint of a given straight line, another line is drawn, and produced to meet the perpendiculars erected at the ends of the given line, the triangles so formed are congruent.



Ex. 7. If two straight lines bisect each other and their extremities are joined, how many pairs of congruent triangles are formed? Prove your statement.

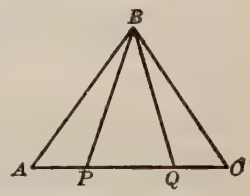
Ex. 8. If, on the legs of an isosceles triangle, equal segments are laid off from the base, and lines are drawn from the extremities of the segments to the opposite vertices, prove that two pairs of congruent triangles are formed.



Ex. 9. On the straight line AD, $AB=CD$. Also $AF=DH$, and $CF=BH$. Prove the triangles AFC and BHD congruent.

Ex. 10. Two right triangles are congruent if their corresponding legs are equal.

Ex. 11. In the isosceles triangle ABC, $AB=BC$, $AQ=PC$. Prove the triangles ABP and QBC congruent.



Ex. 12. The altitudes from the extremities of the base of an isosceles triangle upon the legs of the triangle divide the figure into how many pairs of congruent triangles? Prove this.

Ex. 13. In a given quadrilateral, two adjacent sides are equal and a diagonal bisects the angle between these sides. Prove that the diagonal bisects the quadrilateral.

Ex. 14. If, from the ends of the shorter base of an isosceles trapezoid, lines are drawn parallel to the legs and produced to meet the other base, prove that a pair of congruent triangles is formed.

173. Proving lines equal.—Lines (segments) may be proved equal by proving that they are:

(1) *Corresponding parts of congruent triangles;* or

(2) *Opposite equal angles in a triangle;* or

(3) *The opposite sides of a parallelogram;* or

(4) *Parallel lines comprehended between parallel lines.* (See §§ 122, 153, 155.)

174. Demonstration by analysis.—It is often an advantage to use the following method in obtaining the proof of a theorem:

Assume the proposed theorem as true;

Observe what other relations among the parts of the figure must then be true;

Proceed backward thus, step by step, till the required theorem is found to depend on some known truth or truths;

Then, starting with what is known, reverse the steps taken and thus build up a direct proof of the required theorem.

This method is called **solution by analysis.**

We shall now give an example of solution by analysis.

Observe that in the process which follows, we first write the conclusion (that is the last step of the proof) in its proper place; then, above the last step, we write the next to the last step; and so proceed backward and upward toward the first step.

Thus, in the statement given below (as the first stage of the proof), we first write $AD=DB$.

Then, if this is true, we infer that $\triangle ADC$ may be congruent with $\triangle DBC$. Hence we next write $\triangle ADC \cong \triangle DBC$ (above the step $AD=DB$).

As a means of proving $\triangle ADC \cong \triangle DBC$, we know that $CD=CD$ by identity. Hence we write $CD=CD$ above the step $\triangle ADC \cong \triangle DBC$. Since we know that $CD=CD$ is true, we set down the reason, "Ident." in the right-hand column.

Hence the proof as thus far worked out will appear as follows:

Ex. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base.

Given $\triangle ABC$ with $AC=BC$; and $\angle ACB$ bisected by CD.

To prove $AD=DB$.

Proof (First Stage)

$CD=CD$.	**Ident.**
$\triangle ADC \cong \triangle DBC$.	
$AD=DB$.	

After all the steps of the proof have been written by working backward to the first, the proof assumes the following form:

(Second Stage of the Proof)

Proof.	$AC = BC.$	Hyp.
	$\angle x = \angle y.$	Hyp.
	$CD = CD.$	Ident.
	$\triangle ADC \cong \triangle DBC.$	
	$AD = DB.$	Corr. sides of congruent $\triangle$ are $=$.

We next number the steps (and reasons) in direct order 1, 2, 3, etc., filling in such reasons as were necessarily omitted in the process of working backward. We thus obtain the

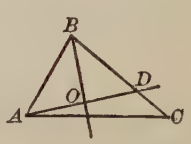
(Third or Final Stage of the Proof)

Proof.	1.	$AC = BC.$	1. Hyp.
	2.	$\angle x = \angle y.$	2. Hyp.
	3.	$CD = CD.$	3. Ident.
	4.	$\therefore \triangle ADC \cong \triangle DBC.$	4. § 81.
	5.	$AD = DB.$	5. Corr. sides of congruent $\triangle$ are $=$. **Q.E.D.**

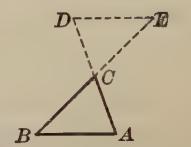
EXERCISES: GROUP 20

Equality of Lines

Ex. 1. Given ABC any $\triangle$, BO the bisector of $\angle ABC$, and $AD \perp BO$. Prove $AO = OD$.



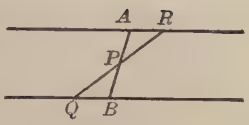
Ex. 2. If two sides of a triangle are produced, each its own length, through the common vertex, the line joining the extremities of the produced parts equals the third side of the triangle (*i.e.*, $DE = BA$).



Ex. 3. If, at any point in the bisector of an angle, a perpendicular is erected to the bisector and produced to meet the sides of the angle, the perpendicular is divided into two equal parts at the given point.

Ex. 4. If, on the legs of an isosceles triangle, equal segments are laid off from the base, lines drawn from the ends of these segments to the opposite vertices are equal.

Ex. 5. Given APB and RPQ straight lines, $AR \parallel QB$, and $AP = PB$. Prove $RP = PQ$.



Ex. 6. Draw an angle and on its sides lay off equal segments from the vertex. Join the extremities of the segments thus formed with any point in the bisector of the angle. Prove that these two joining lines are equal.

Ex. 7. The altitudes of an isosceles triangle upon the legs are equal.

Ex. 8. The diagonals of a rectangle are equal.

Ex. 9. The medians of an isosceles triangle to the legs are equal.

Ex. 10. If two altitudes of a triangle are equal, the triangle is isosceles.

Ex. 11. The perpendiculars to a diagonal of a parallelogram from a pair of opposite vertices are equal.

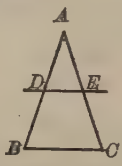
Ex. 12. If the equal sides of an isosceles triangle are produced through the vertex so that the produced parts are equal, the lines joining the extremities of the produced parts to the extremities of the base are equal.

Ex. 13. If the base of an isosceles triangle is trisected, lines drawn from the vertex to the points of trisection are equal.

Ex. 14.

In the bisectors of the equal angles of an isosceles triangle, the segments next to the base are equal ($AO = OP$).

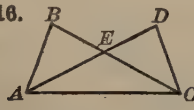
Ex. 15.



Given $\triangle ABC$, $AB = AC$, and $DE \parallel BC$.

Prove $AD = AE$.

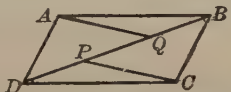
Ex. 16.



Given $AB = DC$, $BC = AD$.

Prove $AE = EC$.

Ex. 17.



Given $ABCD$ a parallelogram and $PB = DQ$.

Prove $AQ = PC$.

175. Proving angles equal.—Among the ways of proving that two angles are equal are the following:

(1) *The given angles are corresponding parts of congruent triangles;* or

(2) *They are opposite equal sides in an isosceles triangle;* or

(3) *They are vertical angles;* or

(4) *They are complements* (*or supplements*) *of equal angles;* or

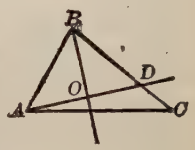
(5) *By use of the properties of parallel lines;* or

(6) *Their corresponding sides are parallel, or perpendicular.*

(See §§ 66, 69, 84, 101, 102, 118, 120.)

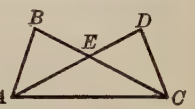
EXERCISES: GROUP 21

EQUALITY OF ANGLES



Ex. 1. Given ABC any $\triangle$, BO the bisector of the $\angle ABC$, and $AD \perp BO$. Prove $\angle BAO = \angle BDO$.

Ex. 2. If, at any point in the bisector of an angle, a perpendicular is erected and produced to meet the sides of the angle, the perpendicular makes equal angles with the sides of the angle.



Ex. 3. Given $AB = DC$, and $AD = BC$. Prove $\angle B = \angle D$.

Ex. 4. If, on the legs of an isosceles triangle, equal segments are laid off from the base, the lines drawn from the extremities of the segments to the opposite vertices make equal angles with the base.

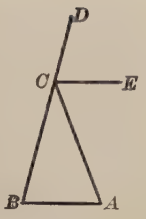
Ex. 5. The altitudes upon the legs of an isosceles triangle make equal angles with the base.

Ex. 6. The diagonals of a rhombus bisect its angles.

Ex. 7. In an isosceles triangle, the exterior angles made by producing the base are equal.

Ex. 8. Given $AC = CB$, and $DE \parallel AB$. Prove $\angle CDE = \angle CED$.

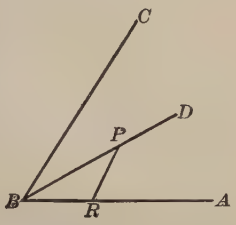
Ex. 9. Given $\angle DCA$ an exterior angle of $\triangle ABC$, $CE \parallel AB$, and $\angle DCE = \angle ECA$. **Prove** $\angle B = \angle A$.



Ex. 10. Conversely, **given** $\angle A = \angle B$, and $CE \parallel AB$. **Prove** that CE bisects $\angle DCA$.

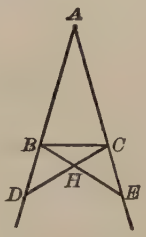
Ex. 11. Given BD the bisector of the angle ABC, and $PR \parallel CB$. **Prove** that PBR is an isosceles $\triangle$.

Let the student state this theorem in general language.



Ex. 12. Given $\triangle ABC$, with AB produced through D, and AC through E, $AB = AC$, and $BD = CE$. **Prove** $\angle BCD = \angle CBE$.

How many pairs of congruent $\triangle$ are there in the figure? Of equal lines? Of equal $\angle$?



Ex. 13. A line drawn through the vertex of an angle, perpendicular to the bisector of the angle, makes equal angles with the sides of the given angle.

Ex. 14. If a straight line which bisects one of two vertical angles is produced, it bisects the other vertical angle also.

176. Proof that two lines are parallel may be obtained by showing that:

(1) *The lines are cut by a transversal so that a pair of alternate interior angles are equal;* or

(2) *A pair of corresponding angles are equal;* or

(3) *Two interior angles on the same side of the transversal are supplementary;* or

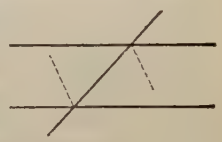
(4) *The lines are opposite sides of a parallelogram.*

(See §§ 93, 96, 99, 147.)

EXERCISES: GROUP 22

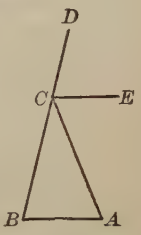
Parallel Lines

Ex. 1. If two sides of a triangle are produced, each its own length, through the common vertex, the line joining their extremities is parallel to the third side of the triangle.



Ex. 2. The bisectors of two alternate interior angles of parallel lines are parallel.

Ex. 3. Given $\angle DCA$ an exterior angle of $\triangle ABC$, $\angle A = \angle B$, and $\angle DCE = \angle ECA$. **Prove** $CE \parallel BA$.



Ex. 4. The bisectors of the opposite angles of a parallelogram are parallel.

Ex. 5. Two straight lines are parallel if two points on one line are equidistant from the other line.

177. The **proof of a numerical property of rectilinear figures** (of Book One) usually depends on one of the following principles:

(1) *The sum of the angles about a given point on the same side of a straight line passing through the point is* 180°.

(2) *The sum of the angles about a point is* 360°.

(3) *The sum of the three angles of a triangle is* 180°.

(4) *The sum of the interior angles of parallel lines on the same side of a transversal is* 180°.

(5) *The sum of the interior angles of a convex polygon of* **n** *sides is* (**n**−2) 180°.

(See §§ 67, 68, 103, 108, 166.)

EXERCISES: GROUP 23

NUMERICAL PROPERTIES

Ex. 1. If an exterior angle of a triangle is 123° and an opposite interior angle is 38°, find the other two interior angles of the triangle.

Ex. 2. Find the angle formed by the bisectors of the two acute angles of a right triangle.

Ex. 3. If two angles of a triangle are 50° and 60°, find the angle formed by their bisectors. If the two angles contain $p°$ and $q°$, find the angle formed by their bisectors.

Ex. 4. If the vertex angle of an isosceles triangle is 40°, and a perpendicular is drawn from an extremity of the base to the opposite side, find the angles of the figure.

Ex. 5. If the vertex angle of an isosceles triangle is 40°, find the angle included between the altitudes drawn from the extremities of the base to the opposite sides.

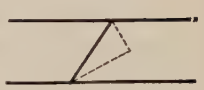
Ex. 6. How many degrees are there in each angle of an equiangular dodecagon? Of an equiangular n-gon?

Ex. 7. How many diagonals are there in a pentagon? In a hexagon? In a decagon? In an n-gon?

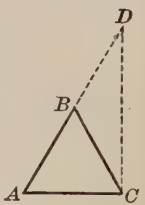
The **methods of proving that a given angle is a right angle** (or **that a given line is perpendicular to another given line**) or **that one angle is the supplement of another,** are closely related to the above methods of obtaining the numerical values of given angles.

Ex. 8. The bisectors of two supplementary adjacent angles form a right angle (are perpendicular).

Ex. 9. The bisectors of two interior angles on the same side of a transversal of two parallel lines form a right angle.



Ex. 10. If one of the legs (AB) of an isosceles triangle is produced its own length (BD), and its extremity (D) is joined to the other end of the base (C), the line last drawn (DC) is perpendicular to the base.



178. Algebraic method of proving theorems.—The proof of certain properties of a geometric figure is often facilitated by *using an algebraic symbol for an unknown angle or an unknown line of the figure, and using an equation or other algebraic method of solution.*

EXERCISES: GROUP 24

Algebraic Method

Ex. 1. Find the number of degrees in an angle which equals three times its complement.

[Sug. See Ex. 24, p. 20.]

Ex. 2. Find the number of degrees in an angle which equals its supplement. In an angle which equals one third of its supplement.

Ex. 3. The angular space about a point is divided into four angles which are in the ratio 1, 2, 3, 4. Find the number of degrees in each angle.

[Sug. $x+2x+3x+4x=360°$, etc.]

Ex. 4. The angles of a triangle are in the ratio 1, 2, 3. Find the angles.

Ex. 5. If one acute angle of a right triangle is double the other acute angle, find the number of degrees in each of these angles.

Ex. 6. If one of the angles of a parallelogram is double another of the angles, find all the angles of the parallelogram.

Ex. 7. One of the base angles of a triangle is double the other, and the exterior angle at the vertex is 105°. Find the angles of the triangle.

Ex. 8. How many sides has a polygon in which the sum of the angles is fourteen right angles?

[Sug. $n-2=7$; find n.]

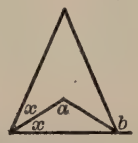
Ex. 9. How many sides has a polygon in which the sum of the angles is ten right angles? Ten straight angles? 720°?

Ex. 10. How many sides has an equiangular polygon one of whose angles is seven fourths of a right angle?

Ex. 11. If the base of any triangle is produced in both directions, the sum of the exterior angles thus formed, diminished by the vertex angle, is equal to a straight angle.

[Sug. $180°-a+180°-b-(180°-a-b)=$, etc.]

Ex. 12. The bisectors of the base angles of an isosceles triangle include an angle which is equal to the exterior angle at the base.

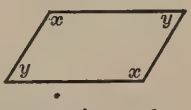


[Sug. To prove $a=b$, denote one of the base $\measuredangle$s by $2x$, etc.]

Ex. 13. In an isosceles triangle, the altitude upon one of the legs makes an angle with the base which equals one half the vertex angle.

[Sug. To prove $a=\frac{1}{2}b$, show that $a=90°-x$, $b=180°-2x$, etc.]

Ex. 14. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.



[Sug. $2x+2y=360°$, etc.]

179. Use of auxiliary lines.—The demonstration of a property of a geometrical figure is frequently facilitated by drawing one or more auxiliary lines on the figure. For examples of the use of such lines, see Props. IV, V, IX, etc., of Book One.

Some of the principal auxiliary lines used on rectilinear figures are:

(1) *A line connecting two given points.*

(2) *A line through a given point parallel to a given line.*

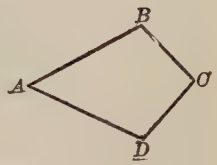
(3) *A line through a given point perpendicular to a given line.*

(4) *A line making a given angle with a given line.*

(5) *A line produced its own length.*

EXERCISES: GROUP 25

Auxiliary Lines

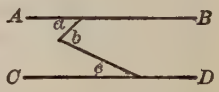


Ex. 1. In the quadrilateral $ABCD$, given $AB = AD$, and $BC = CD$. Prove $\angle B = \angle D$.

[Sug. Draw AC, etc.]

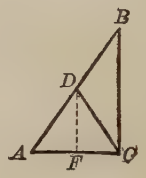
Ex. 2. Prove that the angles at the base of an isosceles trapezoid are equal.

Ex. 3. State and prove the converse of Ex. 2.



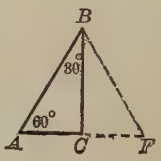
Ex. 4. Given $AB \parallel CD$. Prove $\angle b = \angle a + \angle c$.

Ex. 5. Conversely, given $\angle b = \angle a + \angle c$. Prove $AB \parallel CD$.



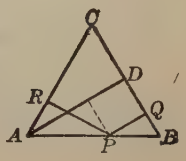
Ex. 6. The median to the hypotenuse of a right triangle is one half the hypotenuse.

[Sug. Draw $DF \parallel BC$. Then $AF = FC$ (§ 168), etc.]



Ex. 7. In a triangle whose angles are 30°, 60°, and 90°, the hypotenuse is twice as long as the shorter leg.

[Sug. Produce AC to F, making $CF = AC$. Then show that $\triangle BCF \cong \triangle ABC$, and $\triangle ABF$ is equilateral, and hence, $AB = AF = 2AC$.]



Ex. 8. In an isosceles triangle, the sum of the perpendiculars drawn from any point in the base to the legs is equal to the altitude upon one of the legs.

In some cases, it is useful to draw *two or more auxiliary lines.*

Ex. 9. If the opposite sides of a hexagon are equal and one pair of sides (AB and CD) are parallel, the opposite angles of the hexagon are equal.

[Sug. Draw AC and BD; use § 159.]

180. Indirect demonstrations.—In Book One, three methods of indirect proof have been used:

1. The **reduction to an absurdity** (reductio ad absurdum), that is, the *proof that the negative of a given theorem leads to an absurdity.* (See Prop. X.)

2. The **method of exclusion,** that is, *showing that any statement inconsistent with the given theorem cannot be true.* This method is a special case of the preceding; the negative of a given theorem is divided into two parts which are separately shown to be impossible. (See Props. XXVII, XXX.)

3. The **method of coincidence,** that is, *proof that a given line coincides with another line, which fulfills certain required conditions.* (See Prop. XV.)

EXERCISES: GROUP 26

Indirect or Negative Demonstrations

Prove the following by an indirect method:

Ex. 1. Every point within an angle and not in the bisector of the angle is unequally distant from the sides of the angle.

[Sug. In the given angle, take P, any point not in the bisector of the angle. Then, if P is not unequally distant from AO and OB, it must be equally distant from them, etc.]

Ex. 2. If, from a point P in a line AB, lines PC and PD are drawn on opposite sides of AB, making the angle APC equal to the angle BPD, PC and PD are in the same straight line.

[Sug. From P draw PQ in the same straight line with PC, and show that PD coincides with it.]

Ex. 3. The bisectors of two vertical angles are in the same straight line.

Ex. 4. In the triangle ABC, D is any point in the side AB, and E is any point in the side AC. Prove that BE and DC cannot bisect each other.

EXERCISES: GROUP 27

Theorems Proved by Various Methods

Ex. 1. If two opposite angles of a quadrilateral are bisected by the diagonal connecting their vertices, the quadrilateral is bisected by this diagonal.

Ex. 2. In a triangle that is not isosceles, the median to the base is produced through the base, and a perpendicular is drawn from each extremity of the base to the median thus produced. Prove that these perpendiculars are equal.

Ex. 3. If the perpendiculars from the extremities of the base of a triangle to the other two sides are equal, (1) these perpendiculars make equal angles with the base, (2) the triangle is isosceles.

Ex. 4. If the lines AB and CD intersect, then $AB+CD>AC+BD$.

Ex. 5. The vertex angle of an isosceles triangle is 44°, and one of the base angles is bisected by a line produced to meet the opposite side. Find all the angles of the figure.

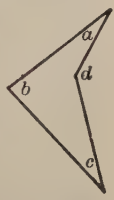
Ex. 6. Any side of a triangle is greater than the difference between the other two sides.

Ex. 7. Perpendiculars drawn from the midpoint of the base of an isosceles triangle to the legs are equal.

Ex. 8. State and prove the converse of Ex. 7.

Ex. 9. In an isosceles triangle, an exterior angle at the base equals a right angle increased by one half the vertex angle.

Ex. 10. Prove that $\angle d = \angle a + \angle b + \angle c$ in the diagram at the right. Is this theorem true when $\angle d$ is a reflex angle?



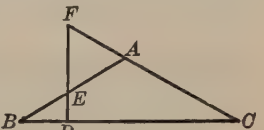
[Sug. Draw an auxiliary line.]

Ex. 11. In the equilateral triangle ABC, BC is produced to D. Prove $BD > AD > AB$.

Ex. 12. If, from a point in the bisector of an oblique angle, lines are drawn parallel to and meeting the sides of the angle, the quadrilateral formed is a rhombus.

Ex. 13. If the median of a triangle is perpendicular to the base, the triangle is isosceles.

Ex. 14. **Given** $AB = AC$, $\angle BAC = 4\angle B$, and $DF \perp BC$. **Prove** $\triangle EFA$ equilateral.



[Analysis. If $\triangle EFA$ is equilateral, $\angle FEA = 60°$. $\therefore \angle DEB = 60°$. $\therefore \angle B = 30°$.

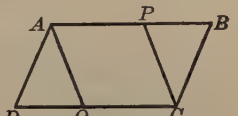
Hence, to get a direct proof, show that $\angle C = 30°$, by using $\angle BAC = 4\angle C$.]

Ex. 15. If the diagonals of a quadrilateral bisect each other at right angles, what kind of figure is the quadrilateral? Prove this.

Ex. 16. From the point in which the altitudes drawn to the legs of an isosceles triangle intersect, a line is drawn to the vertex. Prove that this line bisects the angle at the vertex.

Ex. 17. If, from a point within an acute angle, perpendiculars are drawn to the sides of the angle, the angle formed by these perpendiculars is the supplement of the given angle.

Ex. 18. If, in the parallelogram $ABCD$, $BP = DQ$, then $AQCP$ is a parallelogram.



[Analysis. If $AQCP$ is a $\square$, $AP =$ and $\| QC$. $\therefore$ begin the direct proof by showing that $AP =$ and $\| QC$.]

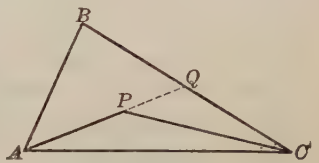
Ex. 19. If the diagonals of a parallelogram are equal, what kind of figure is the parallelogram? Prove this.

Ex. 20. If the angle A of the triangle ABC is 50° and the exterior angle BCD is 120°, which is the longest side of the triangle?

Ex. 21. Two isosceles triangles are congruent if the base and an angle of one are equal to the base and the corresponding angle of the other.

Ex. 22. Two equilateral triangles are congruent if an altitude of one is equal to an altitude of the other.

Ex. 23. If, from a point within a triangle, two lines are drawn to the extremities of one side of the triangle, the sum of the other two sides of the triangle is greater than the sum of the two lines thus drawn. (That is, prove $AB+BC>AP+PC$.)

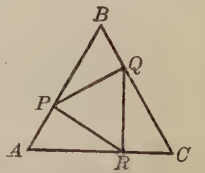


(Ex. 23)

Ex. 24. On the figure of Ex. 23, prove $\angle APC$ greater than $\angle B$.

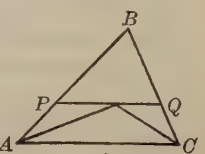
Ex. 25. The hypotenuse of a right triangle is 8 in. and one of the other sides is 4 in. Find the number of degrees in each angle of the triangle. (See Ex. 7, p. 104.)

Ex. 26. If ABC is an equilateral triangle and $AP=BQ=CR$, then PQR is an equilateral triangle.



(Ex. 26)

Ex. 27. The two base angles of a triangle are bisected, and through the point of intersection of the two bisectors a line is drawn parallel to the base. Prove that the part of this line intercepted between the two sides equals the sum of the segments of the sides included between the parallel and the base. (That is, prove $PQ=AP+QC$.)



(Ex. 27)

Ex. 28. The hypotenuse of a right triangle is 10 in. and one of the acute angles is 60°. Find the length of the side opposite the other acute angle.

Ex. 29. The base AB of the triangle ABC is divided into four equal parts by the points D, E, F, in order. P is the midpoint of AC and Q of BC. Prove $PD=QF$.

Ex. 30. The bisectors of the angles of a parallelogram form a rectangle.

Ex. 31. The bisectors of the angles of a rectangle form a square.

Ex. 32. If lines are drawn through the vertices of a quadrilateral parallel to the diagonals, a parallelogram is formed which is twice as large as the original quadrilateral.

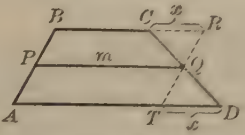
Ex. 33. Lines drawn from two opposite vertices of a parallelogram to the midpoints of a pair of opposite sides trisect a diagonal of the parallelogram.

Ex. 34. On the diagonal AC of a parallelogram $ABCD$, equal parts, AP and CQ, are marked off. Prove $BPDQ$ a parallelogram. How many pairs of congruent triangles does the figure contain?

Ex. 35. The opposite angles of an isosceles trapezoid are supplementary.

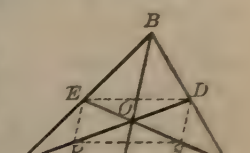
Ex. 36. In an isosceles trapezoid, the diagonals are equal.

Ex. 37. The line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and is equal to one half the sum of the bases.



[Suggestion. Through Q, the midpoint of the leg CD, draw $RT \parallel AB$ and meeting BC produced at R. Then prove $\triangle CQR \cong \triangle TQD$, and $PQ \parallel BR$ and AT, by the method used in Prop. XL, p. 92. Then $m = BR = BC + x$; also $m = AT = AD - x$. $\therefore 2m = BC + AD$, etc.]

Ex. 38. The medians of a triangle meet in a point which is two thirds of the distance from any vertex to the midpoint of the opposite side.



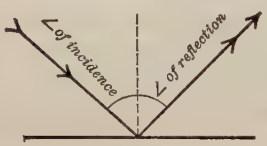
[Suggestion. Let two of the medians, AD and EC, intersect at O, and take R the midpoint of AO, and S the midpoint of CO. Show $ED = \frac{1}{2}AC$ and $\parallel RS$ by § 171. Use §§ 159, 157 to show $AR = RO = OD$, etc. Hence CE crosses AD at a point such that $AO = \frac{2}{3}AD$. In like manner, it may be shown that BF crosses AD at the point O.]

EXERCISES: GROUP 28

Practical Applications

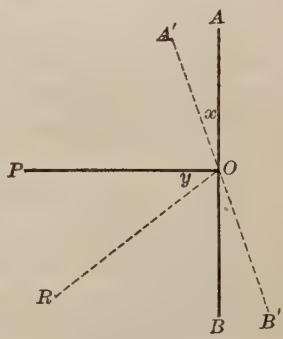
Ex. 1. Draw a figure showing the points of the compass as used by a mariner. On this compass, how many degrees are there between N. and N. N. E.? Also between N. and E. N. E.? Between N. N. E. and S. by E.? Between any two successive compass points?

Ex. 2. When a ray of light strikes a mirror and is reflected, the angle of incidence equals the angle of reflection.

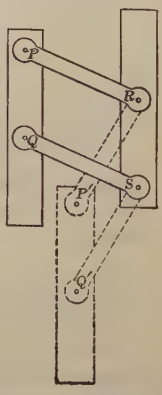


Let AB be a mirror, and PO a ray of light striking the mirror at right angles. Let the mirror be rotated about O to the position $A'B'$. The ray of light PO will then be reflected in the direction OR.

Prove $\angle y = 2\angle x$.



Ex. 3. The diagram shows a drawing instrument called the *parallel rulers*. The dotted outline shows the rest of the instrument in another position, the part RS remaining fixed. The distance $PQ = RS$, $PR = QS$. Hence show that PQ is parallel to RS.



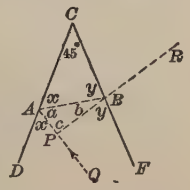
In like manner, show that $P'Q'$ is parallel to RS. Hence show that PQ is parallel to $P'Q'$.

If a line is drawn perpendicular to PQ and another line perpendicular to $P'Q'$, the perpendiculars thus drawn will be parallel to each other. (§ 107, lines perpendicular to parallel lines are parallel.)

Rods combined in these ways are instances of a kind of mechanism, called a linkage, which is of wide importance. Observe, for instance, the system of links which connects the driving wheels of a locomotive with the piston in the cylinder; also the jointed rods connecting the walking beam of a steamboat with the engine. Look up, in the Century Dictionary for instance, the words *linkage*, *cell*, and *parallel motion*.

Ex. 4. In the trusses of steel bridges, why are the beams and rods arranged, as far as possible, so as to form a network of triangles and not of quadrilaterals, or pentagons, for instance? (See Ex. 19, p. 88; also § 87.)

Ex. 5. Let DC and FC be two walls perpendicular to the plane of the paper. Let small mirrors be attached to these walls at A and B. Let $\angle C = 45°$. Let a ray of light pass through Q to A, and be reflected to B and thence to P. Prove that $\angle APB$ is a right angle.



[Sug. Use the principle for the angles of incidence and reflection stated in Ex. 2.

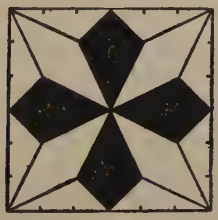
Then in the triangle ABC, $x+y+45° = 180°$, or $x+y = 135°$.

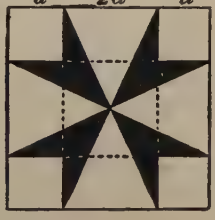
About the points A and B, $2x+a+2y+b = 360°$.

$$\therefore a+b = 90°, \text{ etc.}]$$

The preceding is the principle of an instrument, called the *optical square*, used by foresters in constructing right angles. A ray of light coming from R through a small hole at B above the mirror will make a right angle with the ray coming from Q through P to A.

Ex. 6. By the aid of squared paper, construct the following designs. Extend the last design to include at least five major loops.





Stations	Bearings	Distances
A	N. 30° E.	300 ft.
B	N.	200 ft.
C	S. 67° E.	450 ft.
D	S. 60° W.	600 ft.

Ex. 7. Draw a map for the following survey notes to the scale of 400 ft. to the inch. In laying off the angles, draw a dotted north-and-south line through each station and then use a protractor.

EXERCISES: GROUP 29

Review Work

Make a list of the properties of.

Ex. 1. One straight line (in connection with such points as may be related to the line, as in § 57).

Ex. 2. Two or more parallel lines (in connection with transversals and angles formed by their use).

Ex. 3. Two or more straight lines that are not parallel.

Ex. 4. Angles (independent of triangles or parallel lines).

Ex. 5. A single triangle.

Ex. 6. Two triangles.

Ex. 7. A quadrilateral.

Ex. 8. A parallelogram.

Ex. 9. A rhombus.

Ex. 10. Define as many different kinds of triangles as you can.

Ex. 11. Classify triangles according to two different methods.

Ex. 12. The three sides and the three angles of a triangle are called its six parts. Draw two triangles which have three parts of one triangle equal to three parts of the other, but which are not congruent.

Ex. 13. The efficiency value of Prop. IV (p. 38) is that it enables us to determine, without effort, that two angles of the triangle are equal, if we know that the sides opposite these angles are equal. State the efficiency value of Prop. I. Also of Prop. II. Of Prop. III.

Ex. 14. State the efficiency value of Prop. V. Of Prop. XVII. Of Prop. XIX.

Ex. 15. Give the different methods of proving two lines equal.

Ex. 16. Give the different methods of proving two angles equal.

BOOK TWO

THE CIRCLE

181. A **circle** is a closed curve all points of which lie in the same plane and are equidistant from a point in the plane called the **center.**

Sometimes the word circle is used to mean the portion of a plane inclosed by the curved line described above.

A circle is named by naming its center, as the circle O; or by naming two or more points on it, as the circle ACD.

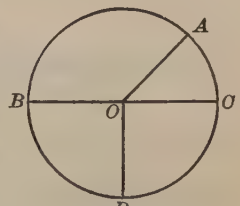


FIG. 1

182. The **circumference** of a circle is the length of the circle.

183. A **radius** of a circle is a straight line drawn from the center to any point on the curve; as OA. A **diameter** is a straight line drawn through the center and terminated by the circle; as BC.

184. An **arc** is any portion of a circle; as AC. A **semicircle** is an arc equal to one half the circle; as BAC.

185. A **chord** is a straight line joining the extremities of an arc; as EF.

Every chord subtends, or *determines*, two arcs. A **minor arc** is an arc less than a semicircle. A **major arc** is an arc larger than a semicircle. Thus, for the chord EF the minor arc is EPF, and the major arc is ETF.

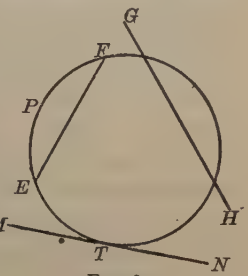


FIG. 2

If the arc determined by a given chord is mentioned, the minor arc is meant, unless otherwise specified.

186. A **tangent** is a straight line which meets a circle and which, however far it may be produced, has but one point in common with the circle; as *MN* (Fig. 2, p. 113).

187. A **secant** is a straight line which cuts a circle and which, if produced far enough, has two points in common with the circle; as *GH* (Fig. 2).

188. A **central angle** is an angle whose vertex is at the center of a circle and whose sides are radii; as the angle *COA* (Fig. 1).

PROPERTIES OF THE CIRCLE INFERRED IMMEDIATELY

189. *Radii of the same circle, or of equal circles, are equal.*

190. *The diameter of a circle equals twice its radius.*

191. *Diameters of the same circle, or of equal circles, are equal.*

192. *If two circles are equal* (i.e., *may be made to coincide*, § 7), *their radii are equal; and conversely.*

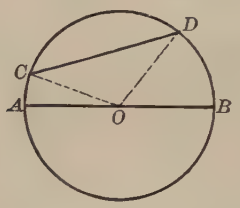
193. *A diameter of a circle bisects the circle.* For, by placing the two parts of the circle so that the diameters coincide and their arcs fall on the same side of the diameter, we see that these arcs coincide. (§ 181.)

194. *A straight line cannot intersect a circle in more than two points.* For, if a straight line can intersect a circle in three or more points, three or more equal lines (radii) can be drawn from the same point (the center) to the straight line. But this is impossible. (§ 143.)

Ex. Draw a diagram (composed of one circle and straight lines) which shall contain all of the geometric objects defined in §§ 181–188. Name these objects in terms of letters.

Proposition I. Theorem

195. *The diameter of a circle is greater than any other chord.*



Given AB a diameter, and CD any other chord in the circle O.

To prove $AB > CD$.

Proof. 1. Draw the radii OC and OD.	1. Post. 1.
2. Then in the $\triangle OCD$, $OC + OD > CD$.	2. Why?
3. $OC = OA$, and $OD = OB$.	3. § 189.
4. Substituting OA for OC, and OB for OD, $OA + OB > CD$; that is, $AB > CD$.	4. Axs. 9, 7. Q.E.D.

Ex. 1. On paper draw a circle with a radius of $1\frac{1}{2}$ in. Draw a diameter of the circle. Cut the circle out of the paper, fold it on the diameter as an axis, and show that the two parts of the circle coincide.

Ex. 2. Draw a circle with any convenient radius. Take any point, A, on the circle, and from it draw a chord which shall be 50 per cent longer than the radius.

Ex. 3. Draw a circle and in it construct a central angle of 60°. Also a central angle of 120°.

Ex. 4. Draw two circles, O and O', each with one inch as radius. In these circles construct central angles AOB and $A'O'B'$, each of 60°. Measure the chords AB and $A'B'$. How do these chords compare in length? Is each one of these chords longer or shorter than a radius of one of the circles?

Proposition II. Theorem

196. *In the same circle, or in equal circles, equal central angles intercept equal arcs.*

Given the equal circles O and O', and the equal central $\measuredangle AOB$ and $A'O'B'$.

To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

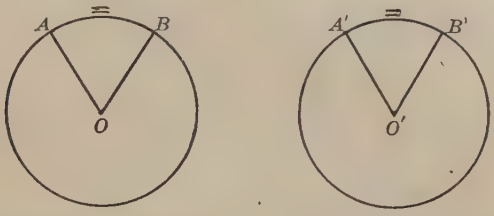
Proof. 1. Apply the circle O' to the circle O so that the center O' coincides with the center O, and the radius $O'A'$ with the radius OA.	1. Geom. Ax. 2.
2. Then $O'B'$ will take the direction of OB.	2. By hyp., $\angle O' = \angle O$.
3. B' will fall on B.	3. $OB = O'B'$, being radii of equal ⓢ. (§ 189.)
4. Hence $\overset{\frown}{AB}$ will coincide with $\overset{\frown}{A'B'}$.	4. All the points on the two arcs are equidistant from the center O. (§ 181.)
5. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$.	5. Geom. Ax. 4. Q.E.D.

Ex. 1. Draw a circle with a radius of one inch. In this circle draw two diameters in such a position with reference to each other that they shall divide the circle into 4 equal arcs.

Ex. 2. Draw a circle and divide it into 8 equal arcs.

PROPOSITION III. THEOREM (CONVERSE OF PROP. II)

197. *In the same circle, or in equal circles, equal arcs determine equal central angles.*



Given the equal circles O and O', and the equal arcs AB and $A'B'$ determining the central $\measuredangle$ O and O'.

To prove $\angle O = \angle O'$.

Proof. 1. Apply the circle O' to the circle O so that the center O' coincides with the center O, and the point A' with the point A.	1. Geom. Ax. 2.
2. Then the point B' will fall on B.	2. $\overset{\frown}{A'B'} = \overset{\frown}{AB}$, by hyp.
3. Hence the radius $O'A'$ will coincide with OA, and the radius $O'B'$ with OB.	3. Only one straight line can be drawn connecting two points. (§ 57.)
4. Hence $\angle O = \angle O'$.	4. Geom. Ax. 4. Q.E.D.

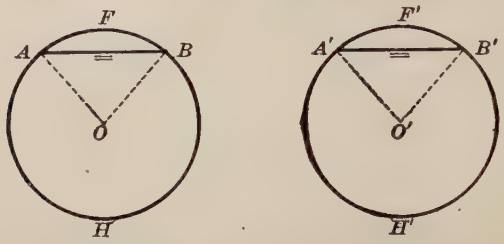
198. COR. *If in the same circle, or in equal circles, two central angles are unequal, the greater angle intercepts the greater arc;* and

CONVERSELY, *if two arcs are unequal, the greater arc determines the greater angle at the center.*

Ex. What is the efficiency value of Prop. II (p. 116)? Of Prop. III?

Proposition IV. Theorem

199. *In the same circle, or in equal circles, equal chords determine equal minor arcs and equal major arcs.*



Given the equal circles O and O', the chord AB = chord $A'B'$, the minor arcs AFB and $A'F'B'$, and the major arcs AHB and $A'H'B'$.

To prove $\overset{\frown}{AFB} = \overset{\frown}{A'F'B'}$ and $\overset{\frown}{AHB} = \overset{\frown}{A'H'B'}$.

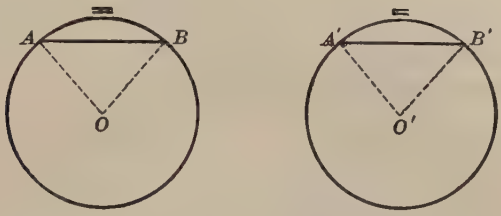
Proof. 1. Draw the radii OA, OB, $O'A'$, $O'B'$.	1. Post. 1.
2. Then, in $\triangle AOB$ and $A'O'B'$, $AB = A'B'$.	2. Why?
3. Also $AO = A'O'$, and $BO = B'O'$.	3. Why?
4. $\therefore \triangle AOB \cong \triangle A'O'B'$.	4. Why?
5. $\therefore \angle O = \angle O'$.	5. Why?
6. $\therefore \overset{\frown}{AFB} = \overset{\frown}{A'F'B'}$.	6. § 196.
7. Circle O = circle O'.	7. Hyp.
8. $\therefore \overset{\frown}{AHB} = \overset{\frown}{A'H'B'}$.	8. Ax. 3. **Q.E.D.**

200. Efficiency value.—The above theorem enables us to find whether two arcs (in the same circle or in equal circles) are equal, by measuring or comparing the two chords which determine the given arcs.

Ex. In the above figure, if chord $AB = 1$ in., chord $A'B' = 1$ in., and arc $AFB = 1\frac{1}{4}$ in., find the length of arc $A'F'B'$.

PROPOSITION V. THEOREM (CONVERSE OF PROP. IV)

201. *In the same circle, or in equal circles, if two arcs are equal, their chords are equal.*



Given the equal circles O and O', and $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove chord AB = chord $A'B'$.

Proof. 1. Draw the radii $OA, OB, O'A', O'B'$.	1. Post. 1.
2. Then, in $\triangle AOB$ and $A'O'B'$, $\angle O = \angle O'$,	2. In the same circle, or in equal circles, equal arcs determine equal central angles. (§ 197.)
3. Also $OA = O'A'$, and $OB = O'B'$.	3. Why?
4. $\therefore \triangle AOB \cong \triangle A'O'B'$.	4. Why?
5. $\therefore AB = A'B'$.	5. Why? **Q.E.D.**

Ex. 1. In the above figure, if arc $AB = 1\frac{1}{4}$ in., arc $A'B' = 1\frac{1}{4}$ in., and chord $AB = 1$ in., find chord $A'B'$ without measuring it.

Ex. 2. What is the efficiency value of Prop. V?

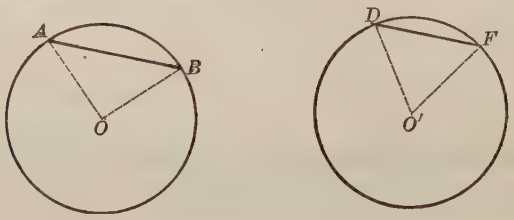
Ex. 3. Draw a circle O with a radius of one inch. In this circle construct central angles of 60° and 120°. Measure the chords which subtend these angles. Is the chord of 120° double the chord of 60°?

Ex. 4. Make up and work an example similar to Ex. 3, using 45° as one of the central angles.

Proposition VI. Theorem

202. I. *If in the same circle, or in equal circles, two minor arcs are unequal, the greater arc determines the greater chord;* and

II. Conversely, *if two chords are unequal, the greater chord determines the greater minor arc.*



I. **Given** the equal circles O and O', and $\overset{\frown}{AB} > \overset{\frown}{DF}$.

To prove chord $AB >$ chord DF.

Proof. 1. Draw the radii OA, OB, $O'D$, $O'F$.	1. Post. 1.
2. Then, in the $\triangle AOB$ and $DO'F$, $OA = O'D$, and $OB = O'F$.	2. § 189.
3. $\overset{\frown}{AB} > \overset{\frown}{DF}$.	3. Hyp.
4. $\angle O > \angle O'$.	4. § 198.
5. $\therefore$ chord $AB >$ chord DF.	5. § 140.

II. Conversely. **Given** the equal circles O and O', and chord $AB >$ chord DF.

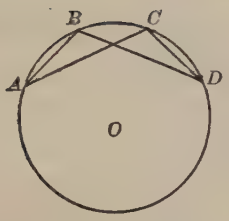
To prove $\overset{\frown}{AB} > \overset{\frown}{DF}$.

Proof. 1. In the $\triangle AOB$ and $DO'F$, $OA = O'D$, and $OB = O'F$.	1. Why?
2. Also $AB > DF$.	2. Why?
3. $\therefore \angle O > \angle O'$.	3. § 141.
4. $\therefore \overset{\frown}{AB} > \overset{\frown}{DF}$.	4. § 198.

Q.E.D.

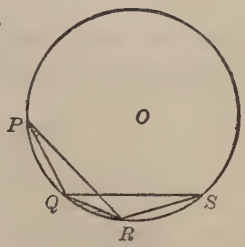
EXERCISES: GROUP 30

Ex. 1.



Given circle O with chord AB = chord CD.
Prove chord AC = chord BD.

Ex. 2.



Given circle O with chord PR = chord QS.
Prove $\triangle PQR \cong \triangle QSR$.

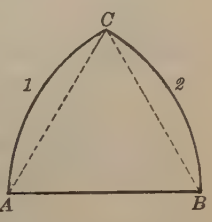
Ex. 3. A, B, C, and D are points taken in succession on a semicircle, and arc AC is greater than arc BD. Prove that chord AB is greater than chord CD.

Ex. 4. State the converse of the theorem in Ex. 3 and prove it.

Ex. 5. AP is a diameter of a semicircle BP and CP are chords such that BP lies between AP and CP. Prove arc $AB <$ arc AC.

Ex. 6. The triangle ABC is equilateral. $A1C$ and $B2C$ are arcs of circles whose centers are B and A, respectively. The arch formed by the arcs is called an equilateral Gothic arch.

Draw a straight line as base and on it construct an equilateral Gothic arch.



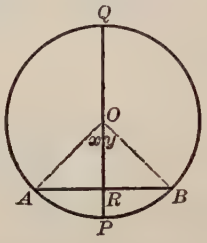
Ex. 7. If a wheel revolves 20 times per minute, through how many degrees does it revolve per second?

Ex. 8. Two circles have a common center. A diameter is drawn in one with extremities A and B. In the other circle, a diameter is drawn with end points X and Y, making an angle with AB. Show that the line joining A and X is parallel to the line joining B and Y.

Ex. 9. On a semicircle, P, Q, R, S are points in succession. Arc PQ = arc RS. The chords PR and QS intersect in H. Prove $QH = HR$.

Proposition VII. Theorem

203. *A diameter perpendicular to a chord of a circle bisects the chord and the arcs determined by the chord.*



Given the circle O, and the diameter $PQ \perp$ chord AB at R.

To prove $AR = BR$, $\overset{\frown}{AP} = \overset{\frown}{BP}$, $\overset{\frown}{AQ} = \overset{\frown}{BQ}$.

Proof. 1. Draw the radii OA and OB.	1. Post. 1.
2. Then, in the rt. $\triangle OAR$ and OBR, $OA = OB$.	2. Why?
3. $OR = OR$.	3. Why?
4. $\therefore \triangle OAR \cong \triangle OBR$.	4. Why?
5. $\therefore AR = BR$, and $\angle x = \angle y$.	5. Why?
6. $\therefore \overset{\frown}{AP} = \overset{\frown}{BP}$.	6. In the same circle, or in equal circles, equal central $\measuredangle$ intercept equal arcs. (§ 196.)
7. But $\overset{\frown}{PAQ} = \overset{\frown}{PBQ}$.	7. A diameter of a circle bisects the circle. (§ 193.)
8. $\therefore \overset{\frown}{AQ} = \overset{\frown}{BQ}$.	8. Ax. 3. **Q.E.D.**

204. Cor. 1. *A diameter which bisects a chord (that is not a diameter) is perpendicular to it.*

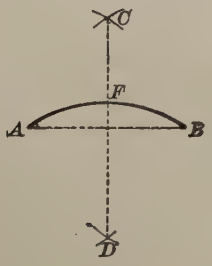
205. Cor. 2. *The perpendicular bisector of a chord passes through the center of the circle, and bisects the arcs determined by the chord.*

206. Cor. 3. *A line which fulfills any two of the following conditions fulfills the other two also:*

(1) *Is bisector of a chord;*
(2) *Is a diameter;*
(3) *Is perpendicular to the chord;*
(4) *Bisects one of the arcs determined by the chord.*

Proposition VIII. Problem

207. *To bisect a given circular arc.*



Given $\overset{\frown}{AB}$.

To bisect $\overset{\frown}{AB}$.

Construction. 1. With A and B as centers, and with convenient equal radii, describe arcs intersecting at C and D.	1. Post. 3.
2. Draw the line CD intersecting $\overset{\frown}{AB}$ at F.	2. Post. 1.
Then $\overset{\frown}{AB}$ is bisected at the point F.	
Proof. 1. Draw the chord AB.	1. Post. 1.
2. $CD \perp$ chord AB at its midpoint.	2. § 128.
3. $\therefore$ CD bisects $\overset{\frown}{AB}$ at F.	3. § 205.
	Q.E.F.

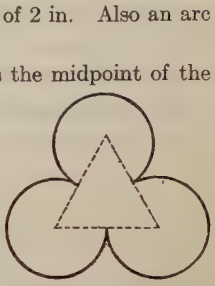
EXERCISES: GROUP 31

Ex. 1. With a radius of 1 in., construct an arc of 90°. Bisect this arc.

Ex. 2. Construct an arc of 60° having a radius of $1\frac{1}{2}$ in. Bisect this arc.

Ex. 3. Construct an arc of $22\frac{1}{2}°$ with a radius of 2 in. Also an arc of 15° with the same radius.

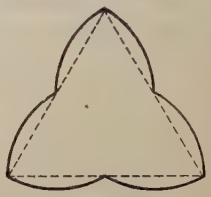
Ex. 4. On the diagram of Prop. VIII, what is the midpoint of the arc AB? Is this also the center of arc AB?



Ex. 5. On the diagram you have drawn for Ex. 1, point out the midpoint of the arc of 90°. Also the center of this arc.

Ex. 6. Construct a trefoil.

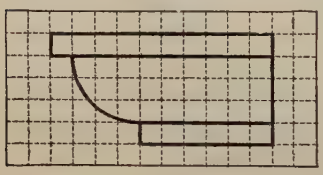
[Sug. Construct an equilateral triangle and bisect one of its sides.]

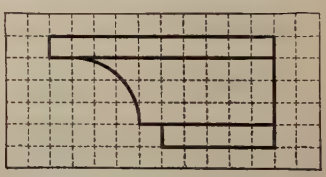


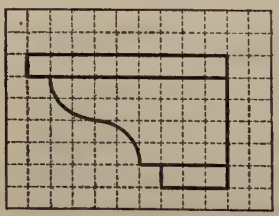
Ex. 7. Construct a quatrefoil by first constructing a square and bisecting one of its sides.

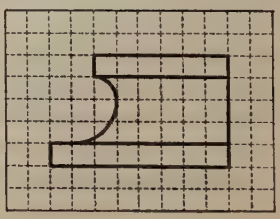
Ex. 8. Construct a figure like the adjoining by first constructing an equilateral triangle and bisecting its sides, and then using the midpoint of each side as a center in constructing arcs.

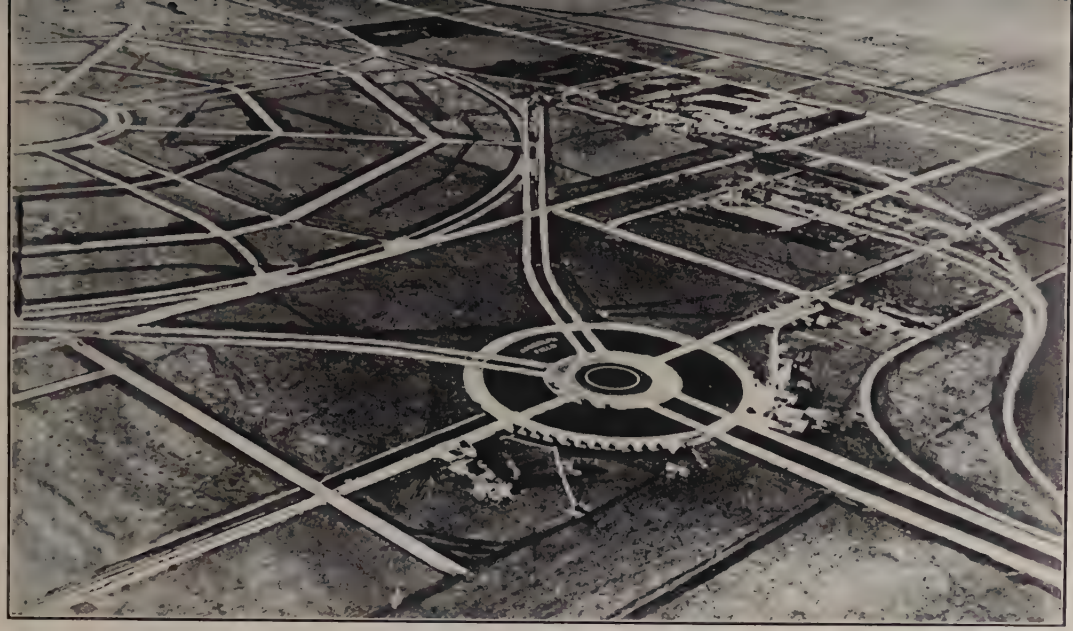
Ex. 9. By the aid of squared paper, construct the following designs for moldings. Note that some curves used are made of two arcs.











THE WEST POINT OF THE AIR

This huge aviation field is Randolph Field, San Antonio, Texas, the new training school for our army air forces. What geometric figures do you recognize in its planning? (*New York World News Service*)

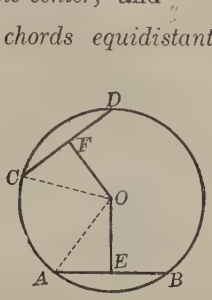
THE YALE BOWL

The old Roman circus, as its name suggests, was a rounded arena for staging the popular spectacles of that day. Now the construction of the modern athletic field is being changed from the square or rectangular plan of several years ago to a circular or elliptical one. What advantage does this change offer to the spectator? (*Airmap Corporation of America*)

Proposition IX. Theorem

208. I. *In the same circle, or in equal circles, equal chords are equidistant from the center;* and

II. Conversely, *chords equidistant from the center are equal.*



I. **Given** the circle O in which the chords AB and CD are equal, $OE \perp AB$, and $OF \perp CD$.

To prove $OE = OF$.

Proof. 1. Draw the radii OA and OC.	1. Post. 1.
2. OE bisects AB, and OF bisects CD.	2. A diameter $\perp$ a chord of a $\odot$ bisects the chord. (§ 203.)
3. In the right $\triangle OAE$ and OCF, $OA = OC$.	3. Why?
4. Also $AE = CF$.	4. Ax. 5.
5. $\therefore \triangle OAE \cong \triangle OCF$.	5. Why?
6. $\therefore OE = OF$.	6. Why?

II. Conversely. **Given** circle O, chords AB and CD, $OE \perp AB$, $OF \perp CD$, and $OE = OF$.

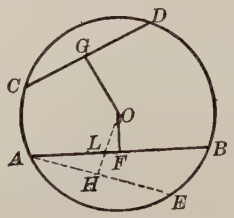
To prove $AB = CD$.

Proof. Let the student supply the proof.

Ex. A wheel revolves 30 times per minute. In how many seconds does a point on its circumference revolve through an arc of 90°? Through an arc of 120°?

Proposition X. Theorem

209. *If in the same circle, or in equal circles, two chords are unequal, the shorter is at the greater distance from the center.*



Given in the circle O the chord $CD <$ chord AB, $OF \perp AB$, $OG \perp CD$.

To prove $OF < OG$.

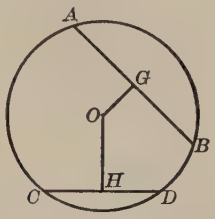
Proof. 1. Chord $AB >$ chord CD.	1. Hyp.
2. $\overset{\frown}{AB} > \overset{\frown}{CD}$.	2. § 202.
3. Mark off on $\overset{\frown}{AB}$ the $\overset{\frown}{AE} = \overset{\frown}{CD}$, and draw the chord AE.	3. Post. 1.
4. $\therefore$ chord $AE =$ chord CD.	4. § 201.
5. Draw $OH \perp AE$ and intersecting AB at L.	5. § 131.
6. Then $OH = OG$.	6. § 208.
7. But $OH > OL$.	7. Ax. 8.
8. Also $OL > OF$.	8. Why?
9. Much more then OH, or its equal OG, $> OF$; that is $OF < OG$.	9. Ineq. Ax. 4. (§ 135.) **Q.E.D.**

Ex. Draw a circle O, with a diameter AB. Construct an angle APB, whose sides, AP and PB, are chords. By use of the protractor, measure angle P. Is this angle greater or less than a right angle?

Also on the same figure, construct another angle, AQB, whose sides are chords. By use of the protractor measure this angle and compare it with angle P.

Proposition XI. Theorem

210. *If in the same circle, or in equal circles, two chords are unequally distant from the center, the one at the greater distance from the center is the shorter.*



Given the chords AB and CD in the circle O, $OG \perp AB$, $OH \perp CD$, $OH > OG$.

To prove chord $AB >$ chord CD.

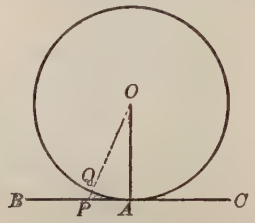
Proof. 1. Either $AB = CD$, or $AB < CD$, or $AB > CD$.	1. Ineq. Ax. 5. (§ 135.)
2. If $AB = CD$, then $OH = OG$.	2. § 208.
3. This is impossible.	3. For $OH > OG$, by hyp.
4. Also, if $AB < CD$, then $OH < OG$.	4. § 209.
5. This also is impossible.	5. Reason 3.
6. $\therefore AB > CD$.	6. Since AB neither $= CD$, nor $< CD$. **Q.E.D.**

Ex. 1. Draw a circle O with any convenient radius. In this circle construct an acute $\angle BAC$ in which BA and AC are chords. Draw the radii OB and OC. By use of the protractor, find the number of degrees in $\measuredangle A$ and O. How many times larger than $\angle A$ is $\angle O$?

Ex. 2. Draw the figure for the following theorem, and state the hypothesis and conclusion in terms of the letters on the figure: "If two equal chords intersect within a circle, the corresponding segments of the chords are equal."

Proposition XII. Theorem

211. *The tangent to a circle at a given point is perpendicular to the radius drawn to that point.*



Given BC tangent to the circle O at the point A.

To prove $BC \perp OA$.

Proof. 1. Take P, any point on BC except A, and draw OP.	1. Post. 1.
2. Then P is outside the circle O.	2. A tangent to a circle can have but one point in common with the circle. (§ 186.)
3. $OP >$ radius OQ, or OA.	3. Ax. 8.
4. $\therefore OA \perp BC$; that is, $BC \perp OA$.	4. § 139. **Q.E.D.**

212. Cor. 1. *A line perpendicular to a radius at its outer extremity is tangent to the circle.*

213. Cor. 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

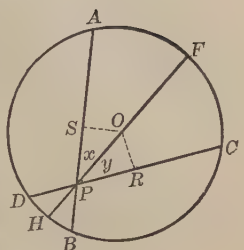
214. Cor. 3. *The perpendicular drawn from the center of a circle to a tangent passes through the point of contact.*

Ex. Construct a circle O, a radius OA, and a line $\perp OA$ at A.

EXERCISES: GROUP 32

Ex. 1. Given the circle O with the equal chords AB and CD intersecting at P. FH is the diameter through P. Prove $\angle x = \angle y$.

[Sug. From the center O, draw OR and $OS \perp$ the chords CD and AB.]



(Ex. 1)

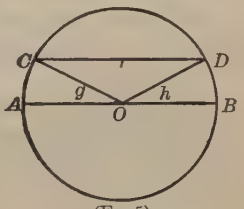
Ex. 2. On a diagram like that on p. 66, with P as a center and with a radius greater than PQ but less than PB, describe a circle. Prove that this circle intercepts equal chords on the lines AB and BC.

Ex. 3. In a given circle, is the chord of an arc of 60° twice as long as the chord of an arc of 30°? Draw a diagram and prove your statement.

Ex. 4. If the perpendiculars from the center upon two chords are equal, the arcs determined by these chords are equal.

Ex. 5. Given the circle O with the radii OC and OD, and AB a diameter; $\angle g = \angle h$. Prove chord $CD \parallel AB$.

[Sug. $\angle COD + \angle C + \angle D = 180°$.]



(Ex. 5)

Ex. 6. In a given circle, tangents drawn at the extremities of a diameter are parallel.

Ex. 7. The line joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

Ex. 8. By the aid of squared paper, construct the following arcs of spirals. (Note that Fig. 1 is composed of semicircles and Fig. 2 of quadrants. See § 234.)

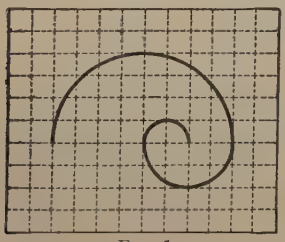


Fig. 1

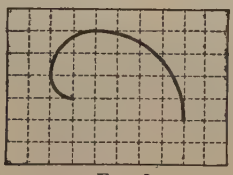
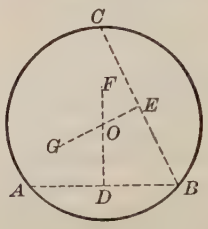


Fig. 2

Proposition XIII. Theorem

215. *Through any three points, not lying in a straight line, one circle, and only one, can be drawn.*



Given A, B, and C any three points not in the same straight line.

To prove that one circle, and only one, can be drawn through A, B, and C.

Proof. 1. Draw the lines AB and BC.	1. Post. 1.
2. Construct DF and EG, the $\perp$ bisectors of AB and BC.	2. § 130.
3. DF and EG must intersect at some point O.	3. § 106.
4. Point O is equidistant from A and B.	4. § 127.
5. Also O is equidistant from B and C.	5. Why?
6. Hence O is equidistant from A, B, C.	6. Ax. 1.
7. Hence, if a circle is described with O as a center and OA as a radius, it will pass through A, B, and C.	7. § 181.
8. But DF and EG can intersect at but one point.	8. § 55.
9. Also between O and B only one straight line can be drawn.	9. § 57.
10. Hence only one circle can be drawn through A, B, and C.	10. With one center and one radius, only one $\odot$ can be drawn.

Q.E.D.

216. Cor. *The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the three vertices.*

217. Efficiency value.—A value of Prop. XIII is that it enables us to determine a whole circle when only three of its points are given or known.

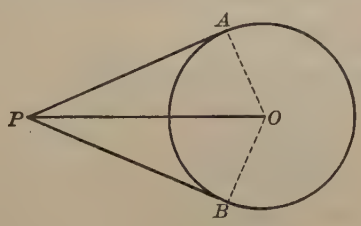
Ex. 1. How many circles can be passed through four given points in a plane, each circle passing through three, and only three, of the given points?

Ex. 2. Through two given points how many circles can be drawn? Draw three of these circles.

Ex. 3. P, Q, R, and S are points taken in succession on a circle so that arc PQ = arc QR = arc RS. Prove that $\angle PQR = \angle QRS$.

Proposition XIV. Theorem

218. *If tangents to a circle from an external point are drawn, they make equal angles with the line joining the given point to the center, and their segments from the given point to the points of contact are equal.*



Given PA and PB tangent to the circle O at the points A and B, respectively.

To prove $\angle APO = \angle BPO$ and $PA = PB$.

Proof. Let the student supply the proof.

Proposition XV. Theorem

219. *Two parallel secants, or two parallel tangents, or a tangent and a secant parallel to it, intercept equal arcs on a circle.*

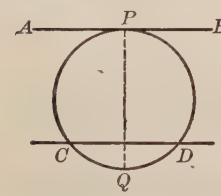


Fig. 1

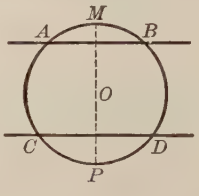


Fig. 2

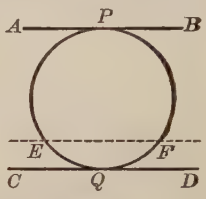


Fig. 3

Case I. **Given** AB (Fig. 1) tangent to the $\odot PCD$ at P, CD a secant $\parallel AB$ and intersecting the circle in C and D.

To prove $\overset{\frown}{PC} = \overset{\frown}{PD}$.

Proof.		
1. Draw the diameter PQ.		1. Post. 1.
2. Then	$PQ \perp AB$.	2. § 211.
3. Also	$PQ \perp CD$.	3. § 105.
4.	$\therefore \overset{\frown}{PC} = \overset{\frown}{PD}$.	4. § 203.

Case II. **Given** AB, and CD (Fig. 2) $\parallel$ secants intersecting the circle in A and B, and C and D, respectively.

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Proof.		
1. Draw the diameter $MP \perp CD$.		1. § 131.
2.	$MP \perp AB$.	2. § 105.
3.	$\therefore \overset{\frown}{AP} = \overset{\frown}{BP}$.	3. § 203.
4.	$\overset{\frown}{CP} = \overset{\frown}{DP}$.	4. Why?
5.	$\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$.	5. Ax. 3

Case III. **Given** AB and CD (Fig. 3) $\parallel$ tangents touching the circle at P and Q, respectively.

To prove $\overset{\frown}{PEQ} = \overset{\frown}{PFQ}$.

Proof. Let the student supply the proof.

220. Cor. *The straight line which joins the points of contact of two parallel tangents to a given circle is a diameter.*

221. The **line of centers** of two circles is the line-segment joining their centers; as OO' (Fig. 1).

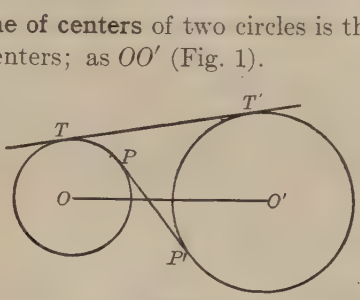


Fig. 1

Sometimes the line of centers is regarded as the indefinite line passing through the two centers.

222. Two circles tangent to each other are circles which are tangent to the same straight line at the same point. They are *tangent externally* or *internally* according as one circle lies entirely without or entirely within the other. See Figs. 2 and 3.

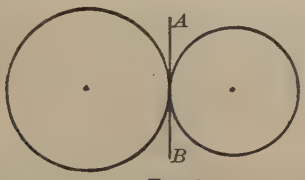


Fig. 2

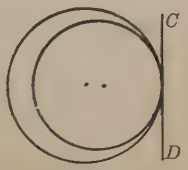


Fig. 3

223. Concentric circles are circles which have the same center. Let the student draw a pair of concentric circles.

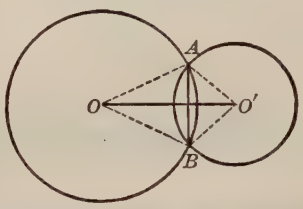
224. Two circles which do not meet may have four common tangents.

A **common internal tangent** of two circles is a tangent which cuts their line of centers; as PP' (Fig. 1) or AB (Fig. 2).

A **common external tangent** of two circles is a tangent which does not cut their line of centers; as TT' (Fig. 1, p. 133) or CD (Fig. 3).

Proposition XVI. Theorem

225. *If two circles intersect each other, the straight line through their centers is the perpendicular bisector of their common chord.*



Given the circles O and O', intersecting at the points A and B.

To prove that OO' is the $\perp$ bisector of the chord AB.

Proof. 1. Draw the radii OA, OB, $O'A$, $O'B$.	1. Post. 1.
2. Then $OA=OB$, and $O'A=O'B$.	2. Why?
3. Hence O and O' are two points equidistant from A and B, and OO' is the $\perp$ bisector of AB.	3. Why? Q.E.D.

226. Cor. *If two circles are tangent to each other, the line through their centers, if extended far enough, passes through the point of contact.*

Ex. 1. Draw two intersecting circles and show that the line of centers is less than the sum of the radii of the two circles.

Ex. 2. Draw two circles in which the line of centers equals the sum of the radii.

Ex. 3. Draw two circles in which the line of centers is greater than the sum of the radii.

227. **A polygon inscribed in a circle** is a polygon all of whose vertices lie in the circle; as $ABCDE$ (Fig. 1).

A **circle circumscribed about a polygon** is a circle which passes through every vertex of the polygon.

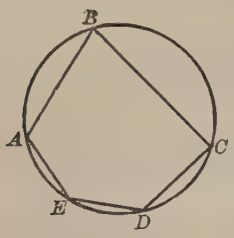


FIG. 1

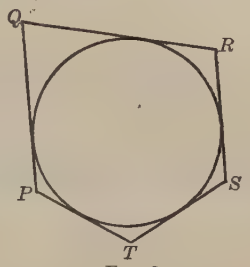


FIG. 2

228. **A polygon circumscribed about a circle** is a polygon all of whose sides are tangent to the circle; as $PQRST$ (Fig. 2).

A circle inscribed in a polygon is a circle to which all the sides of the polygon are tangent.

How many common internal tangents and how many common external tangents have two circles:

Ex. 1. If they touch externally? Touch internally?

Ex. 2. If they intersect?

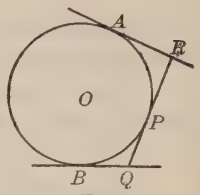
Ex. 3. If one circle lies wholly inside the other?

EXERCISES: GROUP 33

Ex. 1. PA and PB are tangents drawn from the point P to a circle whose center is O. Prove that PO is the perpendicular bisector of the chord AB.

Ex. 2. If two tangents to a circle include an angle of 60° at their point of intersection, the chord joining the points of contact forms, with the tangents, an equilateral triangle.

Ex. 3. Given RA, RQ, and QB tangents of the circle O. Prove $RQ = RA + QB$.



(Ex. 3)

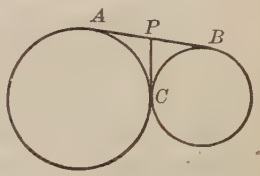
Ex. 4. If a quadrilateral is circumscribed about a circle, show that the sum of one pair of opposite sides equals the sum of the other pair.

Ex. 5. If a hexagon is circumscribed about a circle, show that the sum of three alternate sides equals the sum of the other three sides.

Ex. 6. If a polygon of $2n$ sides is circumscribed about a circle, the sum of n alternate sides equals the sum of the other n sides.

Ex. 7. A parallelogram circumscribed about a circle is equilateral.

Ex. 8. If two circles are tangent externally, the common internal tangent bisects the common external tangent. (That is, prove $PA = PB$.)

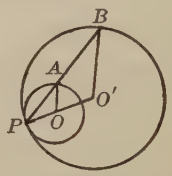


(Ex. 8)

Ex. 9. If two circles are tangent (either externally or internally), tangents drawn to them from any point in the common tangent are equal.

Ex. 10. Two circles whose centers are O and O' are tangent internally at P. The line PAB is drawn intersecting the circles at A and B. Prove that OA and $O'B$ are parallel.

(Ex. 9)

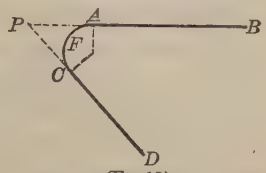


(Ex. 10)

Ex. 11. A, B, and C are points on a circle such that the arcs AB, BC, and AC are all equal. Prove that the angle B contains 60°.

Ex. 12. DF and DH are two equal chords in a circle whose center is O. Prove that DO bisects the angle FDH.

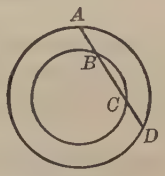
Ex. 13. Two straight street railway tracks, AB and CD, if produced, would meet at the point P. They are, however, connected by the track AFC, which is the arc of a circle to which AB is tangent at A and CD at C. If $PA = 160$ yd., how long is PC? Show how to find the center of the arc AFC.



(Ex. 13)

Ex. 14. Two straight street railway tracks, if produced, would meet at right angles. These tracks are to be connected by the arc of a circle whose radius is 150 yd. Using the scale 1 in. = 100 yd., draw a diagram showing the straight tracks and the connecting curve.

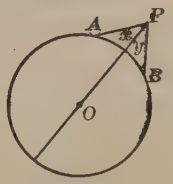
Ex. 15. A straight line intersects two concentric circles. Show that the segments of the line intercepted between the circles are equal. (That is, prove $AB = CD$.)



(Ex. 15)

Ex. 16. AM is a diameter and A, B, C, and M are points in succession on a circle. Prove that chord AC is greater than chord AB.

Ex. 17. Given P any point outside the circle O, A and B points on the circle such that $PA = PB$. **Prove** $\angle x = \angle y$.

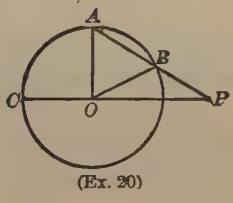


(Ex. 17)

Ex. 18. In Ex. 17, if PA is produced to meet the circle at F, and PB produced to meet the circle at H, prove $AF = BH$.

Ex. 19. In two unequal intersecting circles, LS is a chord of the larger circle perpendicular to the line of centers and intersecting the smaller circle at M and R. Prove $LM = RS$.

Ex. 20. Given O the center of a circle, PBA and POC secants, and BP = the radius. **Prove** $\angle AOC = 3\angle P$.



(Ex. 20)

MEASUREMENT: RATIO

229. Measurement.—For many purposes, the most advantageous way of dealing with a given magnitude is to take a certain definite part of the magnitude as a unit, and then determine the number of times this unit must be taken in order to make up the given magnitude. Ease and precision in dealing with magnitudes are thus obtained.

Geometric magnitudes thus far have been treated as wholes, the object being simply to determine whether two given magnitudes are equal or unequal, or have some other relation. Hereafter, geometric magnitudes will frequently be treated as if composed of units.

To measure a given magnitude is to find how many times the given magnitude contains another magnitude of the same kind taken as a unit.

230. The **numerical measure** of a magnitude is the number which expresses how many times the unit of measure is contained in the given magnitude.

The **ratio** of two magnitudes is the quotient, or indicated quotient, obtained by dividing the first magnitude by the second.

Thus the ratio of 3 ft. to 1 ft. 4 in. is $\frac{36 \text{ in.}}{16 \text{ in.}}$, or $\frac{9}{4}$.

231. Proportion.—When two ratios are equal to each other, they are said to form a proportion.

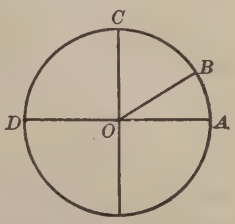
Thus $\frac{8}{12} = \frac{2}{3}$ is a proportion, and the numbers 8, 12, 2, and 3 are said to be in proportion.

232. *Two magnitudes of the same kind have the same ratio as their numerical measures.*

Also it follows from the properties of fractions that *equimultiples of two quantities have the same ratio as the quantities themselves.*

233. A **degree of arc** is $\frac{1}{360}$ part of a circle. (See § 20.)

Hence, if the central angle AOB contains 30 degrees of angle, the intercepted arc AB contains 30 degrees of arc.



234. A **quadrant** is one fourth of a circle; as AC. Hence a quadrant contains 90 degrees of arc.

Also a semicircle contains two quadrants, or 180 degrees of arc.

Ex. 1. With a radius of $\frac{3}{4}$ in., draw a circle, and divide it into four equal arcs.

Ex. 2. With a radius of 1 in., draw a circle, and divide it into eight equal arcs. How many degrees are there in each of these arcs?

Ex. 3. Draw a circle and divide it into six equal arcs. Into twelve equal arcs. How many degrees are in each of these arcs?

Ex. 4. Is 1° of angle always the same size wherever found?

Ex. 5. Is 1° of the arc in Ex. 1 the same size as 1° of the arc in Ex. 2?

Ex. 6. Draw two circles of such a size that 1° of arc in one of them shall equal 1° of arc in the other. Draw another circle in which 1° of arc is smaller than in the two circles first drawn.

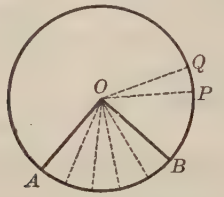
Ex. 7. AB is the diameter of a semicircle, and K and R are points on the semicircle such that the chord KB is greater than the chord RB. Prove the chord AK less than the chord AR.

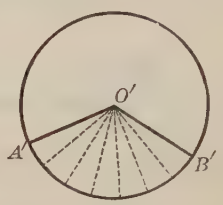
Ex. 8. In a circle O, OH is a radius perpendicular to the diameter CD. P is a point on the arc CH and Q a point on the arc HD such that $\angle POC = \angle QOD$. Prove arc PH = arc QH.

Ex. 9. AD and BC are diameters in the circle O. Prove the chords AB and CD equal and parallel.

Proposition XVII. Theorem

235. *In the same circle, or in equal circles, two central angles are proportional to their intercepted arcs.*





Given the equal ⓢ O and O', with the central $\measuredangle AOB$ and $A'O'B'$ intercepting the arcs AB and $A'B'$.

To prove $\dfrac{\angle AOB}{\angle A'O'B'} = \dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}$.

Proof. 1. Let $\overset{\frown}{PQ}$ (from circle O) be an arc contained in $\overset{\frown}{AB}$ an exact number of times, as m times, and in $\overset{\frown}{A'B'}$ n times.	1. Two magnitudes of the same kind have the same ratio as their numerical measures. (§ 232.)
Then $\dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \dfrac{m\overset{\frown}{PQ}}{n\overset{\frown}{PQ}} = \dfrac{m}{n}$.	
2. From O and O' draw radii to the several points of division of arcs AB and $A'B'$.	2. Post. 1.
3. Then $\angle AOB$ will be divided into m, and $\angle A'O'B'$ into n small angles, all equal.	3. In the same circle, or in equal circles, equal arcs determine equal central angles. (§ 197.)
4. $\dfrac{\angle AOB}{\angle A'O'B'} = \dfrac{m\angle POQ}{n\angle POQ} = \dfrac{m}{n}$.	4. § 232.
5. $\therefore \dfrac{\angle AOB}{\angle A'O'B'} = \dfrac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}$.	5. Ax. 1.
	Q.E.D

If, in the proof of this theorem, the two arcs AB and $A'B'$ should be found to have no common unit of measure, like arc PQ, the theorem may still be proved true by a method of proof called the *method of limits*, which is beyond the scope of this book.

236. Cor. *The number of angular degrees in a central angle is the same as the number of circular degrees in the intercepted arc;* that is, *a central angle is measured by its intercepted arc.*

237. Symbol for measurement; properties.—The expression $\angle AOB \overset{m}{=} \overset{\frown}{AB}$ is read, "The angle AOB is measured by the arc AB." Another form of statement is the following: "The number of degrees in angle AOB equals the numerical measure of arc AB."

Hence the above expression is a form of equality, and the properties of equals, as stated in Axs. 2, 3, 4, 5 (p. 22) and others, apply here also.

Ex. 1. What is the ratio of a quadrant to a semicircle?

Ex. 2. What is the ratio of an angle of an equilateral triangle to one of the acute angles of an isosceles right triangle?

Ex. 3. Draw two circles so that the center of each circle lies on the other circle. Find the ratio of the two arcs in each circle.

Ex. 4. Draw three circles so that the center of each circle lies on the other two circles.

[Sug. First draw an equilateral triangle.]

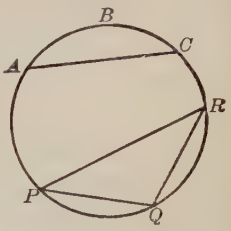
Ex. 5. Draw three circles, each of which shall be tangent to the other two.

Ex. 6. Draw two concentric circles and a line which is a tangent to one of these circles and a chord of the other.

Ex. 7. Draw two concentric circles and a line which is a secant of one and a chord of the other.

238. A **segment** of a circle is a figure formed by an arc of a circle and its chord; as ABC.

Equal segments are segments in the same circle, or in equal circles, having equal chords and arcs.



239. An **inscribed angle** is an angle whose vertex is in the circle and whose sides are chords; as $\angle PRQ$.

240. An **angle inscribed in a segment** is an angle whose vertex is in the arc of the segment and whose sides are chords drawn from the vertex to the extremities of the arc. Thus $\angle PQR$ is inscribed in segment PQR.

Ex. On the above diagram, draw the chords AR and PC intersecting in T. Name all the inscribed angles on the figure. Is ATP an inscribed angle?

Proposition XVIII. Theorem

241. *An inscribed angle is measured by half the intercepted arc.*

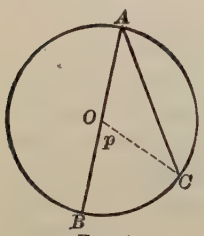


Fig. 1

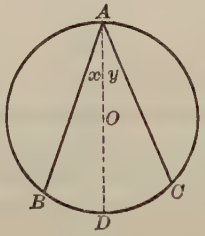


Fig. 2

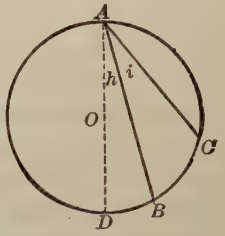


Fig. 3

Case I. *When the center of the circle lies in one side of the inscribed angle.*

Given $\angle BAC$ (Fig. 1) inscribed in the circle O, and AB passing through the center O.

To prove $\angle BAC \overset{m}{=} \frac{1}{2}\overset{\frown}{BC}$.

Proof. 1. Draw radius OC.	1. Post. 1.
2. Then in $\triangle OAC$, $OA = OC$.	2. Why?
3. $\therefore \angle A = \angle C$.	3. Why?
4. But $\angle p = \angle A + \angle C$.	4. Why?
5. $\therefore \angle p = 2\angle A$.	5. Ax. 9.
6. But $\angle p \overset{m}{=} \overset{\frown}{BC}$.	6. § 236.
7. $\therefore 2\angle A \overset{m}{=} \overset{\frown}{BC}$.	7. Ax. 1.
8. $\therefore \angle A \overset{m}{=} \frac{1}{2}\overset{\frown}{BC}$.	8. Ax. 5.

CASE II. *When the center of the circle lies within the inscribed angle.*

Given the inscribed $\angle BAC$ (Fig. 2), with the center of the circle O lying within the angle.

To prove $\angle BAC \overset{m}{=} \frac{1}{2}\overset{\frown}{BC}$.

Proof. 1. Draw the diameter AD.	1. Post. 1.
2. Then $\angle BAD \overset{m}{=} \frac{1}{2}\overset{\frown}{BD}$.	2. Case I.
3. Also $\angle DAC \overset{m}{=} \frac{1}{2}\overset{\frown}{DC}$.	3. Why?
4. $\therefore \angle BAC \overset{m}{=} \frac{1}{2}\overset{\frown}{BC}$.	4. Ax. 2.

CASE III. *When the center of the circle is outside the inscribed angle.*

Given the inscribed $\angle BAC$ (Fig. 3), with the center O outside the angle.

To prove $\angle BAC \overset{m}{=} \frac{1}{2}\overset{\frown}{BC}$.

Proof. Let the student supply the proof.

242. Efficiency value.—By use of the above theorem, if the number of degrees in the intercepted arc is known, the number of degrees in the inscribed angle can be determined immediately.

243. Cor. 1. *All angles inscribed in the same segment, or in equal segments, are equal.*

Thus $\angle A$, A', A'' (Fig. 4) are all equal, for each of them is measured by half the arc BDC.

Also, *in the same circle or in equal circles, angles measured by halves of equal arcs are equal;*

Arcs which measure equal angles are equal.

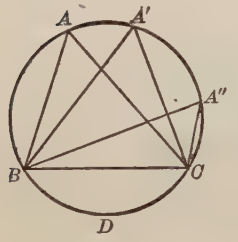


Fig. 4

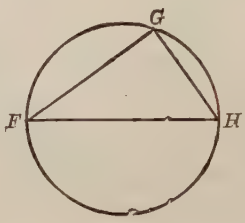


Fig. 5

Fig. 6

244. Cor. 2. *An angle inscribed in a semicircle is a right angle,* for it is measured by half a semicircle.

Thus if FH (Fig. 5) is a diameter, $\angle FGH$ is a right angle.

245. Cor. 3. *An angle inscribed in a segment whose arc is greater than a semicircle is an acute angle. An angle inscribed in a segment whose arc is less than a semicircle is an obtuse angle.*

EXERCISES: GROUP 34

Ex. 1. If in Fig. 1 (p. 142), $\angle A$ contains 23°, how many degrees are there in $\overset{\frown}{BC}$? In $\overset{\frown}{AC}$?

Ex. 2. If in Fig. 1, $\overset{\frown}{AC} = \frac{1}{3}$ of the circle, how many degrees are there in $\angle BOC$? In $\angle BAC$?

Ex. 3. If in Fig. 3, $\overset{\frown}{AC} = 2\overset{\frown}{BC}$, and $\overset{\frown}{BC} = 2\overset{\frown}{DB}$, find $\angle DAC$. Also $\angle DAB$.

Ex. 4. The angle BAC is inscribed in a circle, the arc AB is $\frac{3}{8}$ of the circle, and the arc AC is $\frac{2}{5}$ of the circle. Find $\angle A$.

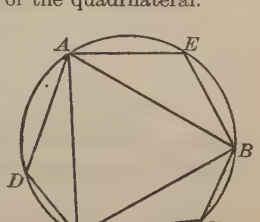
Ex. 5. $\angle BAC$ is inscribed in a circle, $\overset{\frown}{AB}$ is twice $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$ is three times $\overset{\frown}{BC}$. Find $\angle A$.

Ex. 6. The quadrilateral $ABCD$ is inscribed in a circle. The arcs AB, BC, CD are 62°, 84°, 110° respectively. Find each angle of the quadrilateral.

Ex. 7. The quadrilateral $ABCD$ is inscribed in a circle. $\overset{\frown}{AB}=48°$ and $\overset{\frown}{BC}=92°$. $\overset{\frown}{AD}=4\overset{\frown}{DC}$. Find each angle of the quadrilateral.

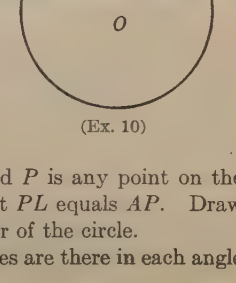
Ex. 8. Prove that each pair of opposite angles in an inscribed quadrilateral are supplementary.

Ex. 9. ABC is an inscribed equilateral triangle. $\measuredangle E$, F, D are inscribed in the segments AEB, BFC, and CDA, respectively. Find the number of degrees in the sum of the $\measuredangle E$, F, D.



(Ex. 9)

Ex. 10. $PQRS$ is a quadrilateral inscribed in the circle O. $\overset{\frown}{PQ}=62°$, $\overset{\frown}{QR}=43°$, and the major arc $RS=212°$. Find all the angles of the figure. Draw the radius OQ and then find all the new angles formed on the figure.



(Ex. 10)

Ex. 11. Draw a circle containing an inscribed quadrilateral. In each of the segments whose chords are the sides of the quadrilateral inscribe an angle. Find the number of degrees in the sum of these inscribed angles.

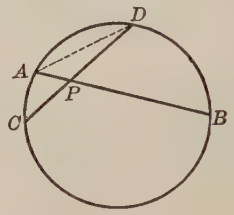
Ex. 12. AB is the diameter of a circle, and P is any point on the circle. AP is produced to the point L so that PL equals AP. Draw PB and prove that BL is equal to the diameter of the circle.

If the angle A contains 22°, how many degrees are there in each angle of the figure?

Ex. 13. ABC is an isosceles triangle inscribed in a circle. The vertex angle B is 30°, and D is the midpoint in the arc BC. If the line AD is drawn, how many degrees are there in the angle DAB?

Proposition XIX. Theorem

246. *An angle formed by two intersecting chords of a circle is measured by half the sum of the intercepted arcs.*



Given the chords AB and CD in the $\odot ADBC$, intersecting within the circle at the point P.

To prove $\angle DPB \overset{m}{=} \frac{1}{2}(\overset{\frown}{DB}+\overset{\frown}{AC})$.

Proof. 1. Draw the chord AD.	1. Post. 1.
2. In $\triangle ADP$, $\angle DPB = \angle A + \angle D$.	2. Why?
3. But $\angle A \overset{m}{=} \frac{1}{2}\overset{\frown}{DB}$.	3. An inscribed angle is measured by half the intercepted arc. (§ 241.)
4. $\angle D \overset{m}{=} \frac{1}{2}\overset{\frown}{AC}$.	4. Why?
5. $\therefore \angle DPB \overset{m}{=} \frac{1}{2}(\overset{\frown}{DB}+\overset{\frown}{AC})$.	5. Ax. 9. **Q.E.D.**

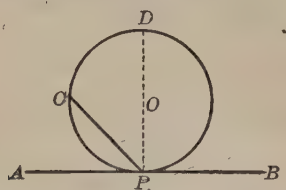
Ex. 1. In the above figure, if arc DB contains 94° and arc AC contains 38°, how many degrees are there in $\angle APC$? In $\angle APD$?

Ex. 2. If arc $AD = 52°$ and $\angle APD = 124°$, find arc CB.

Ex. 3. If in Fig. 1 (p. 142) arc AC contains 112°, how many degrees are there in $\angle A$?

Proposition XX. Theorem

247. *An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.*



Given AB a tangent to the circle O at the point P; PC a chord.

To prove $\angle APC \overset{m}{=} \frac{1}{2}\overset{\frown}{PC}$.

Proof. 1. Draw the radius OP and extend it to form the diameter DP.	1. Posts. 1, 2.
2. $\overset{\frown}{PCD}$ is a semicircle, or 180°.	2. § 193.
3. $\frac{1}{2}\overset{\frown}{PCD} = 90°$.	3. Ax. 5.
4. $\angle APD$ is a rt. $\angle$, or 90°.	4. § 211.
5. $\therefore \angle APD \overset{m}{=} \frac{1}{2}\overset{\frown}{PCD}$.	5. Ax. 1.
6. But. $\angle CPD \overset{m}{=} \frac{1}{2}\overset{\frown}{CD}$.	6. Why?
7. Hence. $\angle APC \overset{m}{=} \frac{1}{2}(\overset{\frown}{PCD} - \overset{\frown}{CD})$, or $\frac{1}{2}\overset{\frown}{PC}$.	7. Ax. 3.
	Q.E.D.

Ex. 1. In the above figure, if arc PC contains 94°, how many degrees are there in $\angle APC$?

Ex. 2. If arc $CD = 80°$, find all the angles on the figure.

Ex. 3. If arc PDC exceeds arc PC by 128°, how many degrees are in each angle of the figure?

Ex. 4. The parallel sides of an inscribed trapezoid determine arcs of 110° and 100° respectively. Find the number of degrees in each angle of the trapezoid.

Proposition XXI. Theorem

248. *An angle formed by two secants of a circle, or by two tangents, or by a secant and a tangent, intersecting at a point outside the circle, is measured by half the difference between the intercepted arcs.*

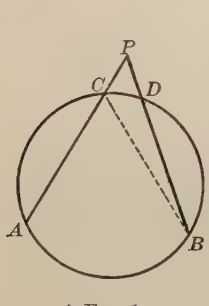


Fig. 1

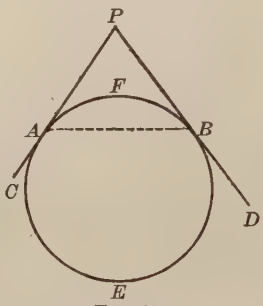


Fig. 2

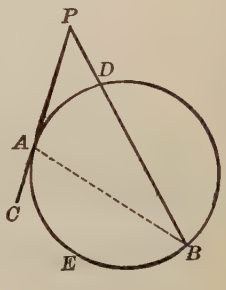


Fig. 3

I. **Given** the $\odot ACDB$ (Fig. 1), and the $\angle APB$ formed by the secants PA and PB, meeting at the point P outside the circle, and intersecting the circle at C and D, respectively.

To prove $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{CD})$.

Proof. 1. Draw the chord CB.		1. Post. 1.
2. Then	$\angle ACB = \angle P + \angle B$.	2. Why?
3.	$\therefore \angle P = \angle ACB - \angle B$.	3. Ax. 3.
4. But	$\angle ACB \overset{m}{=} \frac{1}{2}\overset{\frown}{AB}$.	4. Why?
5. Also	$\angle B \overset{m}{=} \frac{1}{2}\overset{\frown}{CD}$.	5. Why?
6.	$\therefore \angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{CD})$.	6. Ax. 9.

II. **Given** $\angle APB$ (Fig. 2) formed by the tangents PC and PD touching the $\odot AEB$ at A and B, respectively.

To prove $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AEB} - \overset{\frown}{AFB})$.

Proof. Let the student supply the proof.

III. Given the $\odot ADB$ (Fig. 3), and $\angle CPB$ formed by the tangent PC touching the circle at A and the secant PB intersecting the circle at D.

To prove $\angle P \overset{m}{=} \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{AD})$.

Proof. Let the student supply the proof.

Ex. 1. If in Fig. 1 (p. 148) $\overset{\frown}{CD} = 34°$ and $\overset{\frown}{AB} = 108°$, draw AD and find all the angles of the figure.

Ex. 2. If in Fig. 2 (p. 148) angle $P = 80°$, find arcs AFB and AEB.

EXERCISES: GROUP 35

Ex. 1. What new methods of proving two lines equal have been found in Book Two?

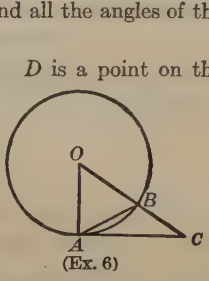
Ex. 2. Two secants to a circle meet and form an angle of 38°. The smaller of the two arcs intercepted by them is 65°. Find the number of degrees in the larger arc.

Ex. 3. $ABCD$ is a quadrilateral inscribed in a circle. $\overset{\frown}{AB} = 68°$, $\overset{\frown}{BC} = 78°$, $\overset{\frown}{CD} = 84°$. Draw the diagonals and find all the angles of the quadrilateral.

Ex. 4. AB and AC are tangents to a circle. D is a point on the major arc CB. $\angle BDC = 44°$. Find $\angle A$.

Ex. 5. The chord CD intersects the diameter AB at the point P. $\overset{\frown}{DB} = 72°$. $\angle DPB = 52°$. Find $\overset{\frown}{BC}$.

Ex. 6. Given O the center of a circle and AC a tangent. Prove $\angle BAC = \frac{1}{2}\angle O$.

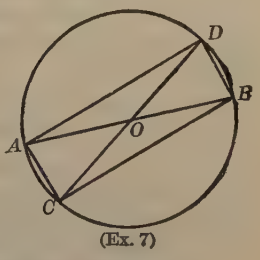


(Ex. 6)

Ex. 7. If AB and CD are two diameters in the circle O, prove $ADBC$ a rectangle.

Ex. 8. If two chords in a circle intersect within the circle at right angles, the sum of a pair of alternate arcs equals a semicircle.

Ex. 9. A chord forms equal angles with the tangents at its extremities.



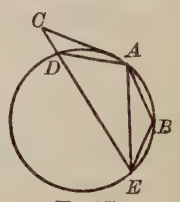
(Ex. 7)

Ex. 10. If one side of an inscribed quadrilateral is produced, the exterior angle so formed equals the opposite interior angle of the quadrilateral.

Ex 11. If an isosceles triangle is inscribed in a circle, the tangent at its vertex makes equal angles with two of its sides.

Ex. 12. A parallelogram inscribed in a circle is a rectangle.

Ex. 13. If A, B, C, D, and E are points taken in succession on a circle, and the arcs AB, BC, CD, and DE are equal, prove that the angles ABC, BCD, and CDE are equal.



(Ex. 15)

Ex. 14. An inscribed angle formed by a diameter and a chord has its intercepted arc bisected by a radius which is parallel to the chord.

Ex. 15. Given AC a tangent and $AB \parallel CE$. Prove $\triangle ACD$ and ABE mutually equiangular.

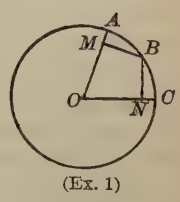
Ex. 16. AB and AC are unequal chords in a circle. D and E are midpoints of the arcs AB and AC, respectively, D and C being on opposite sides of AB, and E and B on opposite sides of AC. The chord DE intersects AB and AC in F and G, respectively. Prove $AF = AG$.

249. Use of auxiliary lines.—In demonstrating theorems relating to the circle, it is often helpful to draw one or more of the following auxiliary lines: *a radius, a diameter, a chord, a perpendicular from the center upon a chord, an arc, a circle, etc.*

EXERCISES: GROUP 36

Auxiliary Lines

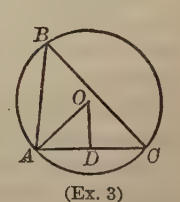
Ex. 1. Given circle O, $\overset{\frown}{AB} = \overset{\frown}{BC}$, $BM \perp OA$, and $BN \perp OC$. Prove $BM = BN$.



(Ex. 1)

Ex. 2. If from any point on a circle a chord and a tangent are drawn, the perpendiculars drawn to them from the midpoint of the arc determined by the chord are equal.

Ex. 3. Given O the center of a circle, and $OD \perp$ chord AC. Prove $\angle AOD = \angle B$.



(Ex. 3)

Ex. 4. From the extremity of a diameter, chords are drawn making equal angles with the diameter. Prove that these chords are equal.

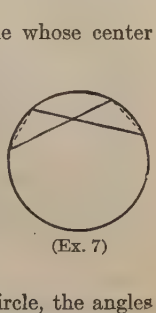
[Sug. Draw $\perp$s from the center to the chords.]

Is the converse of this theorem true?

Ex. 5. If through any point within a circle equal chords are drawn, show that the line drawn from the center to the point of intersection of the chords bisects their angle of intersection.

Ex. 6. Tangents PA and PB are drawn to a circle whose center is O. Prove that angle P equals twice angle OAB.

Ex. 7. If in a circle two equal chords intersect, the segments of one chord equal the segments of the other chord, respectively.



(Ex. 7)

Ex. 8. The chord AB equals the chord CD in a given circle, and the chords, if produced, intersect at the point P. Prove secant PA = secant PC.

[Sug. Draw the chord AC.]

Ex. 9. If a quadrilateral is circumscribed about a circle, the angles at the center determined by a pair of opposite sides are supplementary.

[Sug. Draw radii to the points of contact and show that there are four pairs of equal $\measuredangle$ at the center.]

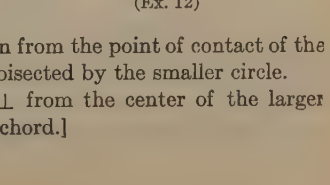
Ex. 10. If an equilateral triangle ABC is inscribed in a circle, and any point P is taken in the arc AB, show that $PC = PA + PB$.

[Sug. On PC take PM equal to PA, draw AM, and prove $\triangle PAB$ and MAC congruent.]

Ex. 11. Two radii perpendicular to each other are produced to intersect a tangent, and from the points of intersection other tangents are drawn to the circle. Prove that the tangents last drawn are parallel.

[Sug. Draw radii to the three points of contact.]

Ex. 12. A common external tangent is drawn to two circles which are exterior to each other. Chords are drawn from the points of tangency to the points where the line of centers, produced through the center of the larger circle, cuts the smaller circle and meets the larger circle. Prove that these chords are parallel.



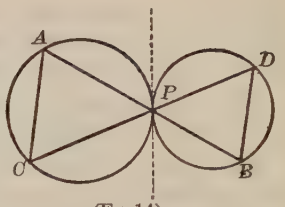
(Ex. 12)

Ex. 13. A circle is described on the radius of another circle as a diameter, and a chord of the larger circle is drawn from the point of contact of the two circles. Prove that this chord is bisected by the smaller circle.

[Sug. If the chord is bisected, a $\perp$ from the center of the larger circle to the chord will also bisect the chord.]

Ex. 14. Two circles are tangent externally at the point P. Through P any two lines APB and CPD are drawn, terminated by the circles. Show that the chords AC and BD are parallel.

[Sug. Draw the common tangent at P. If AC and BD are ||, what $\measuredangle$ must be equal?]



(Ex. 14)

Ex. 15. Two circles intersect at the points P and Q. Lines APB and CQD are drawn, terminated by the circles. Show that AC and BD are parallel.

Ex. 16. If a square is constructed on the hypotenuse of a right triangle, a line drawn from the point where the diagonals of the square intersect to the vertex of the right angle bisects the right angle.

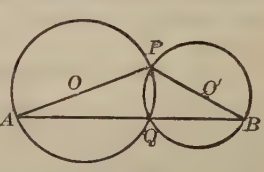
[Sug. Describe a circle on the hypotenuse of the right triangle as a diameter.]

Ex. 17. If two circles are tangent externally at P, and a common tangent touches them at A and B, respectively, the angle APB is a right angle.

Ex. 18. If in the triangle ABC the two altitudes BD and AE are drawn, the angle ABD equals the angle AED.

[Sug. Describe a semicircle on AB as a diameter.]

Ex. 19. Two circles intersect at P and Q. PA and PB are diameters. Prove that QA and QB form a straight line.



(Ex. 19)

Ex. 20. Two equal circles intersect at P and Q; through P, a line is drawn terminated by the circles at A and B. Show that QA equals QB.

[Sug. $\measuredangle A$ and B are measured by what arcs?]

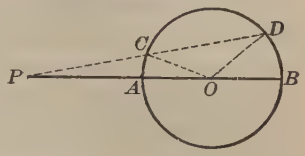
250. A **maximum** is the greatest of a class of magnitudes satisfying certain given conditions, and a **minimum** is the least.

Of the chords in a given circle, which is the maximum?

EXERCISES: GROUP 37

Maxima and Minima

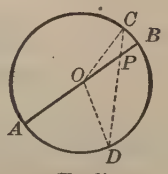
Ex. 1. Of the chords drawn through a given point within a circle, determine which is the greatest, and also which is the least.



(Ex. 2)

Ex. 2. Find the shortest line, and also the longest line, that can be drawn from a given external point to a circle.

[Sug. O being the center, prove in $\triangle PCO$, $PA < PC$; by $\triangle PDO$, $PB > PD$.]



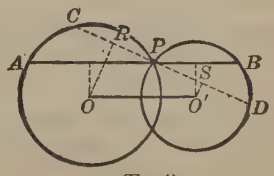
(Ex. 3)

Ex. 3. Find the shortest line, and also the longest line, that can be drawn to a circle from a point within the circle.

[Sug. O being the center, prove by use of $\triangle OPC$, $PB < PC$.]

Ex. 4. If two circles intersect, show that, of lines drawn through a point of intersection and terminated by the circles, that line is a maximum which is parallel to the line of centers.

[Sug. Prove $RS < OO'$.
$\therefore CPD < APB$.]



(Ex. 4)

EXERCISES: GROUP 38

Demonstrations by Indirect Methods

Prove the following by an indirect method (see § 180, p. 105):

Ex. 1. A segment of a circle which contains a right angle is a semicircle.

Ex. 2. If a rectangle is inscribed in a circle, its diagonals are diameters.

Ex. 3. Prove the second part of Prop. VI by an indirect method.

Ex. 4. If the opposite angles of a quadrilateral are supplementary, a circle can be circumscribed about the quadrilateral.

[Sug. Pass a circle through three vertices of the quadrilateral; if it does not pass through the remaining vertex, etc.]

251. A geometrical constant is a geometrical magnitude which may vary in some respect, as in position, but which remains constant in size.

Thus the angles inscribed in a given semicircle vary in position but are all of the same size; viz., a right angle. (See § 244.)

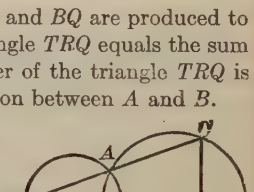
EXERCISES: GROUP 39

Determination of Constants

Ex. 1. AB and AC are tangents to a circle. P is any point on the circle outside the triangle ABC. Prove that, as P moves, the sum of $\angle A$ and $\angle BPC$ is constant.

Ex. 2. In the figure of Ex. 3, p. 136, if AR and BQ are produced to meet at T, show that the perimeter of the triangle TRQ equals the sum of TA and TB; and, hence, that the perimeter of the triangle TRQ is constant, no matter how P may vary in position between A and B.

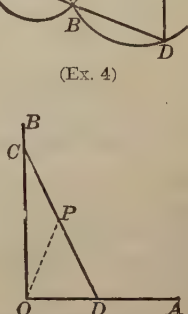
Ex. 3. Show on the figure of Ex. 3, p. 136, that, if O is the center of the circle, $\angle ROQ$ is constant as P varies in position.



(Ex. 4)

Ex. 4. Two circles intersect in the points A and B. From any point P on one circle lines PAC and PBD are drawn, terminated by the other circle. Show that the chord CD is constant.

[Sug. Draw BC and prove $\angle CBD$ a constant.]



(Ex. 5)

Ex. 5. Given $OA \perp OB$, and CD a line of given length moving so that D is always in OA and C in OB, and P the midpoint of CD. Prove that OP is constant in length. (See Ex. 6, p. 104.)

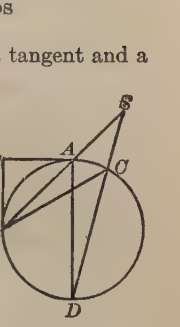
EXERCISES: GROUP 40

Theorems Proved by Various Methods

Ex. 1. The line which bisects the angle formed by a tangent and a chord bisects the intercepted arc also.

Ex. 2. An inscribed trapezoid is isosceles.

Ex. 3. Given TA and TB tangents, arc $AB = 80°$, arc $BD = 95°$, and arc $DC = 150°$; find all the angles of the figure.



(Ex. 3)

Ex. 4. Find the angle formed by the side of an inscribed square and the tangent through the vertex of the square.

Ex. 5. From an external point a secant is drawn through the center of a circle, and also two other secants making equal angles with the first secant. Show that the secants last drawn are equal.

Ex. 6. ABC is a triangle, and on the side AB the point P is taken and on BC the point Q, so that angle BPQ equals angle C. Show that a circle may be circumscribed about the quadrilateral $APQC$.

Ex. 7. Two circles are tangent to each other externally, and a line is drawn through the point of contact and is terminated by the circles. Show that the radii from the extremities of this line are parallel.

Ex. 8. If a circle is described on the leg of an isosceles triangle as diameter, the circle will bisect the base of the triangle.

Ex. 9. The chord of an arc is parallel to the tangent at the mid-point of the arc.

Ex. 10. If two unequal chords intersect in a circle, the greater chord makes the less angle with the diameter through the point of intersection of the chords.

State also the converse of this theorem. Is the converse true?

Ex. 11. The sum of the legs of a right triangle equals the sum of the hypotenuse and the diameter of the inscribed circle.

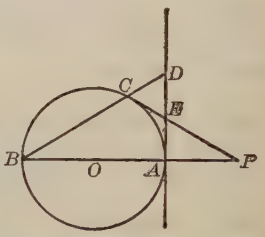
Ex. 12. The sides AB, BC, and AC of a triangle touch the inscribed circle at the points P, Q, and R. Show that angle PQR and one half angle A are complementary.

[Sug. Draw radii from the center O to P and R. Then $\angle PQR = \angle POA$, etc.]

Ex. 13. From the point in which the bisector of an inscribed angle meets the circle, a chord is drawn parallel to one side of the angle. Show that this chord equals the other side of the angle.

Ex. 14. Given AB a diameter, $AP =$ the radius, AD and PC tangents. Prove $\triangle CED$ equilateral.

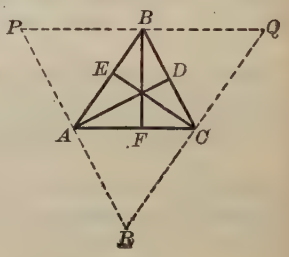
[Sug. Draw OC and CA; then in rt. $\triangle OCP$, $CA =$ radius, $\angle P = 30°$, etc.]



Ex. 15. Perpendiculars are drawn from the extremities of a diameter upon a tangent. Show that the points in which the perpendiculars intersect the tangent are equidistant from the center.

Ex. 16. The altitudes of a triangle meet in a point.

[Sug. Through each vertex of the given $\triangle ABC$, draw a line $\parallel$ the side opposite. Prove PA and AR each $= BC$ by use of § 153; hence AD is the $\perp$ bisector of PR. Use § 216.]

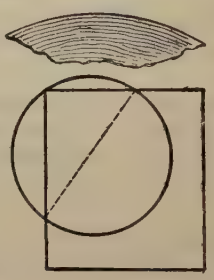


EXERCISES: GROUP 41

Practical Applications

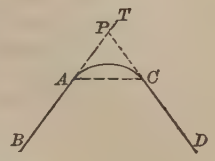
Ex. 1. Given a fragment of a broken wheel, show how to find the radius of the wheel.

Ex. 2. Show how to find the diameter of a given circle by applying a rectangular sheet of paper to the circle. By more than one application, show how to find the center of the circle.



Ex. 3. Show how a pattern maker, by the use of a carpenter's square, can determine whether the cavity made in the edge of a board or piece of metal is a semicircle.

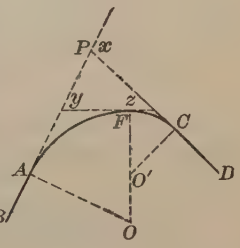
Ex. 4. The curve in a railroad track is usually an arc of a circle tangent to each straight track which it joins. If two straight tracks AB and CD are joined by a circular curve tangent to both of them, and P is the point where AB and CD would meet if extended, prove $\angle TPD = 2\angle TAC$.



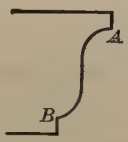
Ex. 5. Let AB and CD be two straight railroad tracks connected by an arc AF of a circle whose center is O, and an arc FC whose center is O', the arcs having a common tangent at F. Prove that the angle of intersection (x) of the two straight tracks, if produced, equals the sum of the central angles of the two tracks which are arcs of circles.

[Sug. $x = y + z$. Use Ex. 4.]

A curve like AFC composed of two or more arcs of different radii is called a *compound curve*. Can you suggest why a compound curve should be used in connecting railroad tracks?

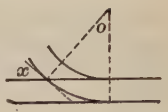


Ex. 6. If the two arcs which compose a compound curve lie on opposite sides of their common tangent, the compound curve is called a *reverse curve*. Thus, on the adjoining figure, AB is a reverse curve. Make a drawing of the adjoining figure, on a scale three times as large as the given figure.



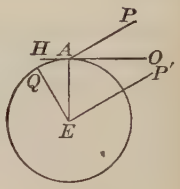
Ex. 7. Construct an isosceles triangle ABC in which the base AB is 1 in. and the altitude is $1\frac{1}{2}$ in. Construct the perpendicular bisector of the side BC and let it meet AB produced at P. Also construct the perpendicular bisector of AC and let it meet AB produced at Q. With P as a center and PB as a radius, construct an arc from B to C. Also with Q as a center and QA as a radius, construct an arc from A to C. The two arcs together will form a Gothic arch.

Ex. 8. What is a railroad frog? If a curved track crosses a straight track, show that the angle of the frog (x) equals the central angle of the curved track (a).



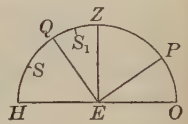
Ex. 9. Prove that the latitude of a place on the earth's surface equals the elevation of the pole. (That is, on the diagram, prove $\angle QEA = \angle PAO$.)

If at a given place the elevation of the north pole of the sky is 41° 30′, what is the latitude of the place?



Ex. 10. Given the sun's declination (*i.e.* distance north or south of the celestial equator), show how to determine the latitude of a place by measuring the zenith distance of the sun. Also by measuring the altitude of the sun above the horizon.

That is, given QS and ZS (or SH), find QZ.

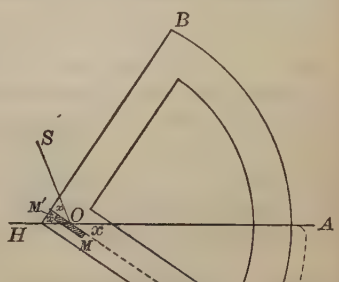


Ex. 11. How was Peary aided by the principles of Exs. 9 and 10 in determining whether he had arrived at the North Pole?

Ex. 12. If on April 6 (the day of the year on which Peary was at the North Pole) the sun was 6° 7′ north of the celestial equator, how high above the horizon should the sun have been, as observed by Peary? At what hour of the day was this?

Ex. 13. The diagram shows an instrument called the *angle meter*. This may be used for measuring either horizontal or vertical angles.

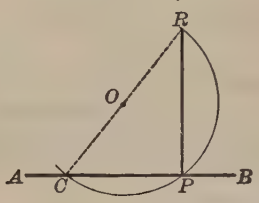
O is the center of the arc BC and MM' is a fixed mirror. For instance, to measure the altitude of the sun the instrument is held vertically, so that a ray SO from the sun when reflected from the mirror passes into the eye (A) in a horizontal direction.



Show that the elevation of the sun above the horizon $= 2\angle x$. The rim BC is graduated so that the angle x can be read on it.

Proposition XXII. Problem

252. *At a given point of a given line, to erect a perpendicular to the line.*



Given the point P in the line AB.

To construct a $\perp$ to line AB at P.

Construction. 1. Take any convenient point O outside the line AB, and with OP as a radius describe a circle intersecting AB at C.	1. Post. 3.
2. Draw CO and produce CO to meet the circle at R.	2. Posts. 1 and 2.
3. Draw RP.	3. Post. 1.
Then RP is the $\perp$ required.	
Proof. Let the student supply the proof.	**Q.E.F.**

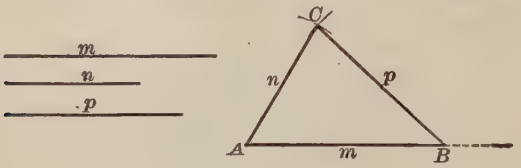
253. Efficiency value.—The above method of erecting a perpendicular to a given line is particularly useful when the point at which the perpendicular is to be drawn is at or near the end of the given line.

Ex. 1. Draw a line 2 inches long, and on it as a side construct a square, making use of the method of § 252 when constructing one line perpendicular to another.

Ex. 2. Using the method of § 252, construct a rectangle 2.5 inches long and 1.5 inches wide.

Proposition XXIII. Problem

254. *To construct a triangle, when the three sides are given.*



Given m, n, and p, the three sides of a triangle.

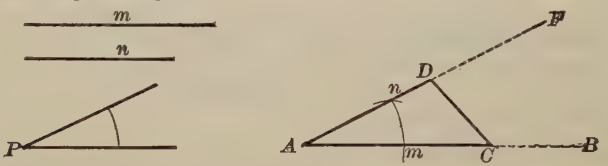
To construct the triangle.

Let the student supply the solution.

Note.—Is the above construction always possible? Give reasons for your statement.

Proposition XXIV. Problem

255. *To construct a triangle, when two sides and the included angle are given.*



Given m and n, two sides of a triangle, and P the angle included by them.

To construct the triangle.

Let the student supply the solution.

Note.—Is the above construction always possible?

Ex. 1. Construct a triangle in which two of the sides are 1 in. and $1\frac{1}{2}$ in., and the included angle is 135°.

Ex. 2. Construct an isosceles triangle in which the base shall be $1\frac{1}{2}$ in. and the altitude 2 in.

Ex. 3. Construct the complement of half a given obtuse angle.

Ex. 4. Construct the supplement of twice a given acute angle.

Ex. 5. How can the figure on p. 91 be constructed with the fewest adjustments of the compasses?

Ex. 6. Draw a line (segment) and mark off three fifths of it.

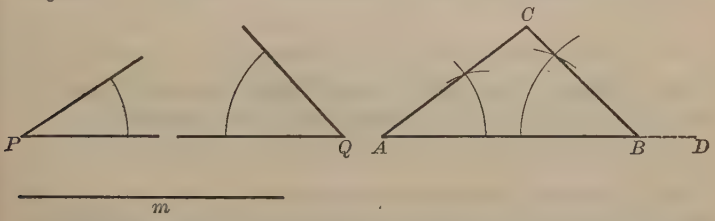
Ex. 7. From a given point on a given circle, how many equal chords can be drawn?

Ex. 8. Through a given point within a given circle, how many equal chords can be drawn?

Ex. 9. From a given point external to a given circle, how many equal secants can be drawn?

Proposition XXV. Problem

256. *To construct a triangle, when two angles and a side are given.*



Given the $\measuredangle P$ and Q and the included side m.

To construct the triangle.

Let the student supply the solution.

Let the student also make the construction when the given side is not included between the given angles.

Note.—Is the above construction always possible? Give numerical illustrations to prove your answer.

Ex. 1. Construct a triangle in which two of the angles are 30° and 45°, and the included side is $1\frac{3}{4}$ in.

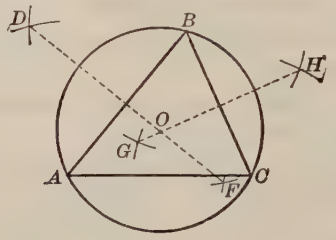
Ex. 2. Construct the complement of half a given angle.

Ex. 3. Construct an angle of 120°; of 150°; of $112\frac{1}{2}$°.

Ex. 4. Trisect a given right angle.

Proposition XXVI. Problem

257. *To circumscribe a circle about a given triangle.*



Given the $\triangle ABC$.

To circumscribe a circle about the $\triangle ABC$.

Let the student supply the solution. (See § 215, p. 130.)

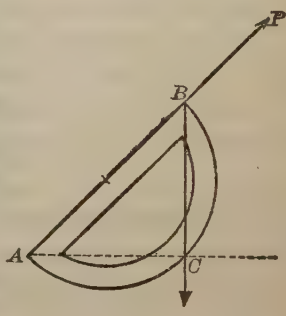
258. The **circumcenter** of a triangle is the center of the circle circumscribed about the given triangle. Thus O in the above figure is the circumcenter of the triangle ABC.

Ex. 1. Draw a triangle whose sides are $2\frac{1}{4}$ in., 2 in., and $1\frac{3}{4}$ in., and circumscribe a circle about it.

Ex. 2. Draw an obtuse triangle and circumscribe a circle about it.

Ex. 3. The line ABP points from the eye of the observer to the top of a tree. ACB is a semicircle ruled like a protractor, with its diameter applied to AP. C is the point where a plumb line from B intersects the outside circle.

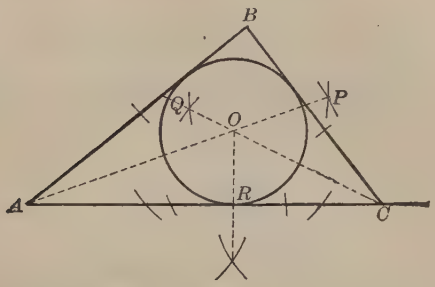
Show that $\angle ACB$ is a right $\angle$. Hence that AC is a horizontal line. Hence that $\angle PAC$, the angle of elevation of P, is measured by $\frac{1}{2}$ arc BC.



Ex. 4. Construct a circle which shall be tangent to a given line at a given point and have a radius of 1 in.

PROPOSITION XXVII. PROBLEM

259. *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

To inscribe a circle in $\triangle ABC$.

Construction. 1. Bisect $\angle BAC$ by line AP, and $\angle BCA$ by line QC.	1. § 88.
2. Then AP and CQ intersect in some point O.	2. § 104.
3. From O, construct $OR \perp AC$.	3. § 131.
4. With O as a center and OR as a radius, describe a circle. This will be the circle required.	4. Post. 3.
Proof. 1. The point O is equidistant from the lines AB and AC.	1. § 129.
2. Also O is equidistant from BC and AC.	2. § 129.
3. $\therefore$ O is equidistant from all three sides of the $\triangle ABC$.	3. Ax. 1.
4. Hence a circle described with O as its center and OR as its radius will touch all the sides of $\triangle ABC$, and therefore be inscribed in $\triangle ABC$.	4. §§ 181, 212, 228.
	Q.E.F.

260. COR. *The bisectors of the angles of a triangle meet in a point.*

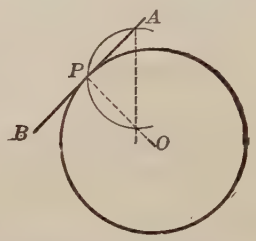
Ex. 1. Construct a right triangle and in it inscribe a circle.

Ex. 2. Construct an obtuse triangle and in it inscribe a circle.

Ex. 3. Write out a complete, formal proof of § 212.

Proposition XXVIII. Problem

261. *To draw a tangent to a circle through a given point on the circle.*



Given P any point on the circle O.

To construct a tangent to the circle through the point P.

Construction. 1. Draw the radius OP.	1. Post. 1.
2. At the point P construct the line $AB \perp OP$.	2. § 252.
Then AB is the tangent required.	
Proof. Let the student supply the proof.	**Q.E.F.**

Ex. 1. Construct a right triangle in which the hypotenuse is 2 in. and one leg is $\frac{3}{4}$ in.

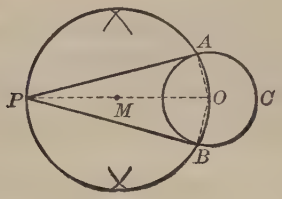
Ex. 2. The hypotenuse of a right triangle is $2\frac{1}{2}$ in. What is the largest altitude which the triangle can have on the hypotenuse?

Ex. 3. Construct a circle O with radius of 2 in., and in it draw two diameters AB and CD $\perp$ each other. Draw BD, and OF, the altitude from O to BD. Prove $\angle OBF = 45°$, and $OF = BF$.

Ex. 4. Construct a circle O with a radius of 2 in. and in it construct a central angle AOB of 120°. Draw the chord AB, and OM the altitude from O to AB. Find the angle AOM in degrees, and the line OM in inches.

Proposition XXIX. Problem

262. *To draw a tangent to a circle through a given external point.*



Given P any point outside the circle O.

To construct through P a tangent to the circle O.

Construction. 1. Draw the line PO.	1. Post. 1.
2. Bisect the line PO at the point M.	2. § 130.
3. With M as a center and MP as a radius, describe a circle intersecting the given circle at the points A and B.	3. Post. 3.
4. Draw PA and PB.	4. Post. 1.
Then PA and PB are the tangents required.	
Proof. 1. $\angle PAO$ is inscribed in a semicircle.	1. Constr.
2. $\therefore \angle PAO$ is a right angle.	2. Why?
3. $\therefore PA$ is tangent to the circle O.	3. Why?
4. In like manner, it may be proved that PB is tangent to the circle O.	4. Reasons 1–3. Q.E.F.

263. The **incenter** of a triangle is the center of the circle inscribed in the triangle, as O in the diagram, p. 163.

Ex. Two equal chords of a given circle are produced till they meet. The angle formed by the extended chords is 18°, and the smaller arc intercepted by them is $\frac{1}{15}$ of the circle. Find each angle of the quadrilateral formed by joining the ends of the two chords

LOCI

264. The **locus of a point** is the path of a point moving according to a given geometric law.

If a point moves in a plane so as to be always two inches distant from a given point, its locus is a circle whose center is the given point, and whose radius is a line two inches in length.

The locus of a point moving so as to be equidistant from two given parallel lines is a straight line lying midway between the two given lines.

The locus of a point may consist of two or more separate lines or parts.

The locus of a point moving so as to be always at a given distance from a given line is two lines, one on either side of the given line, at the given distance from it and parallel to it.

265. To **describe a locus,** state (1) what kind of line, surface, or other geometric object the locus is; then state (2) where it is situated and (3), if possible, give its dimensions.

Thus the locus of a point moving so as always to be two inches from a given point is (1) a circle (2) whose center is the given point and (3) whose radius is two inches.

266. Demonstration of loci.—In order to prove that a given line is the locus of a given point moving according to a given geometric law, it is necessary to prove:

1. *That every point in the given line satisfies the given law or condition.*

2. *That every point which satisfies the given condition lies in the given line.*

Hence the theorem of § 127 may be stated in the following form: *The locus of points equidistant from two given points is the perpendicular bisector of the line joining the two given points.*

267. The theorem of § 129 may also be stated in the following form: *The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.*

EXERCISES: GROUP 42

Ex. 1. Draw exactly the locus of a point moving at the distance of 1 in. from a given point.

Ex. 2. Draw exactly the locus of a point moving at a distance of 1 in. from a given line.

Ex. 3. Draw exactly the locus of a point moving so as to be equidistant from the extremities of a given line one inch long.

Write a description of the following loci (see § 265):

Ex. 4. The locus of the center of a carwheel 2 ft. in diameter, as the wheel rolls on a straight level track.

Ex. 5. The locus of a point which moves so as to be equidistant from two fixed points which are 1 ft. apart.

Determine and describe the following loci. Give no proof.

Ex. 6. Find the locus of a point moving so that it is at a given distance, a, from a given circle whose radius is r.

Ex. 7. Find the locus of the midpoints of the radii of a given circle.

Ex. 8. Find the locus of the midpoints of all chords of a given length drawn in a given circle.

Ex. 9. Find the locus of the vertices of all right triangles having a given hypotenuse as base.

Ex. 10. Find the locus of the midpoints of all the chords drawn from a given point on a given circle.

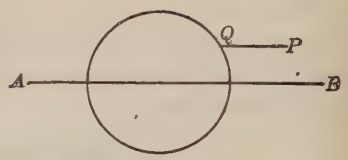
[Sug. Draw a line from the center to the given point, and perpendiculars from the center upon the chords.]

Ex. 11. By use of ruler and compasses, construct a square each side of which is $1\frac{1}{2}$ in. Then construct and describe the locus of a point which moves so as always to be outside the square and $\frac{3}{4}$ in. from it.

Ex. 12. Construct an equilateral triangle whose side is $1\frac{1}{2}$ in., and also the locus of a point moving outside the triangle and at a distance of 1 in. from it. Write a description of this locus.

Ex. 13. In Ex. 5, page 154, find the locus of P.

Ex. 14. QP is a line of given length which moves so that Q is always in a given circle, and QP is always parallel to a fixed line. Find the locus of P.



Ex. 15. Find the locus of the centers of circles which touch a given line at a given point.

[Sug. Construct a number of circles touching the given line at the given point and observe the relation of their centers.]

Ex. 16. Find the locus of the centers of circles with given radius, r, which pass through a given fixed point.

Ex. 17. Find the locus of the centers of circles which touch two given intersecting lines.

Ex. 18. Find the locus of the centers of circles which touch two given parallel lines.

Ex. 19. Find the locus of the centers of circles of given radius, r, which touch a given straight line.

Ex. 20. Find the locus of the centers of circles of given radius, r, which touch a given circle.

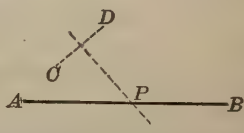
268. Construction of points and of loci.—In constructing a point to meet certain given conditions, it is often helpful to *construct the locus of a point answering one of the given conditions and observe in what point or points it meets a given line, or meets another locus answering another given condition.*

EXERCISES: GROUP 43

Construction of Points and Loci

Ex. 1. Find, in a given line AB, a point P equidistant from two given points C and D.

[Sug. Construct the locus of all points equidistant from C and D, and observe where it intersects the given line AB.]



Ex. 2. Find, on a given circle, a point P which is equidistant from two given points, C and D. (Is it always possible to find such a point?)

Ex. 3. Find, in a given line, a point P which is equidistant from two given intersecting lines.

Ex. 4. Find, in a given line, a point which is at a given distance, d, from a given point.

Ex. 5. Show how to find, on a railroad, a point equidistant from two villages not on the railroad. Is it necessary that the railroad be straight?

Ex. 6. Draw a straight line and locate a point 2 in. from it. By the use of loci, locate the points which are $1\frac{1}{2}$ in. from the given line and at the same distance from the given point.

Ex. 7. Find a point which is at a given distance, a, from a given point, A, and at another given distance, b, from another given point, B. Discuss the limitations of this problem.

Ex. 8. Find a point equidistant from two given points, and at a given distance from a given straight line.

[Sug. Draw the locus of all points equidistant from the two given points, and also the locus of all points at the given distance from the given straight line, etc.]

Ex. 9. Find a point equidistant from two given points, and at a given distance from another given point.

Ex. 10. Find a point equidistant from two given points, and also from two given intersecting lines.

Ex. 11. A pirate's treasure is known to be buried at a place 80 ft. from a given road and equidistant from two given trees. Show how to find the place.

269. Analysis of construction problems.—The method of analysis (see § 174) is of especial value in the solution of construction problems. In general, to investigate the solution of a problem by this method:

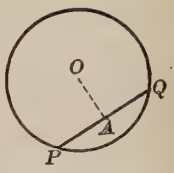
Draw a figure in which the required construction is assumed as made.

Draw auxiliary lines, if necessary.

Observe the relations between the parts of this figure, in order to discover a known relation on which the required construction depends.

Having discovered the required relation, construct another figure by the direct use of this relation.

Ex. Through a given point within a circle, draw a chord which shall be bisected by the given point.



Analysis. Let A be the given point within the given circle O, and let PQ be a chord bisected at the point A. A bisected chord suggests a line OA joining the point of bisection with the center O, and that $OA \perp PQ$ (§ 204).

Synthesis, or Direct Solution. Taking another figure containing the data of the problem, connect the point A with the center of the circle by the line OA.

Through A draw a line $\perp OA$ (§ 252), and meeting the circle at the points P and Q. PQ is the chord required.

EXERCISES: GROUP 44

Construction of Straight Lines

Ex. 1. Draw a line parallel to a given line, and tangent to a given circle.

[Sug. Suppose the required line drawn; then the radius to the point of tangency, if produced, is $\perp$ given line, etc.]

Ex. 2. Draw a line perpendicular to a given line, and tangent to a given circle.

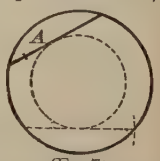
Ex. 3. From two points on a circle, draw two equal and parallel chords.

Ex. 4. Through a given point, draw a line which shall make a given angle with a given line.

Ex. 5. Through a given point, draw a line which shall make equal angles with the sides of a given angle.

[Sug. The bisector of the given $\angle$ will be $\perp$ the required line, etc.]

Ex. 6. Through a given point between two given parallel lines, draw a line of given length with its extremities in the two parallel lines.



(Ex. 7)

Ex. 7. Through a given point, A, within a circle, draw a chord equal to a given line.

Ex. 8. From a given point on a circle, draw a chord at a given distance from the center.

Ex. 9. Through a given point on a circle, draw a chord which shall be bisected by another given chord.

[Sug. Draw the radius to the given point and on it as a diameter describe a circle, etc. When is the solution impossible?]

EXERCISES: GROUP 45

CONSTRUCTION OF RECTILINEAR FIGURES

Construct:

Ex. 1. An equilateral triangle, given the altitude.

Ex. 2. An isosceles triangle, given the base and the altitude.

Ex. 3. An isosceles triangle, given the base and an angle at the base.

Ex. 4. An isosceles triangle, given the vertex angle and the altitude.

Ex. 5. A right triangle, given a leg and the acute angle adjacent.

Ex. 6. A right triangle, given a leg and the acute angle opposite.

Ex. 7. A triangle, given the altitude and the sides including the vertex angle.

[SUG. Through the foot of the altitude draw a line $\perp$ altitude and of indefinite length.]

Ex. 8. A triangle, given two sides and the altitude upon one of them.

Ex. 9. A rhombus, given one angle and the diagonal passing through the vertex of the given angle.

Ex. 10. A parallelogram, given a side, the altitude upon that side, and an angle.

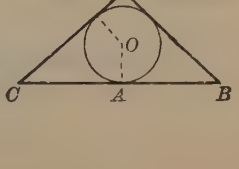
Ex. 11. A parallelogram, given the diagonals and an angle included by them.

Ex. 12. A quadrilateral, given the sides and one angle.

270. Use of auxiliary lines in constructing rectilinear figures.—In constructing polygons, auxiliary lines are frequently of service. Thus it is often of especial value to *construct, first, either the inscribed or the circumscribed circle, and afterward the required triangle or quadrilateral.*

Ex. Construct an isosceles triangle, given the base, b, and the radius, r, of the inscribed circle.

CONSTRUCTION. Draw a circle O with radius equal to r. Draw a tangent at any point A. On this tangent mark off AB and AC each equal $\frac{1}{2}b$. From B and C draw tangents BF and CF to the circle. Then BCF is the required triangle.



EXERCISES: GROUP 46

CONSTRUCTIONS; AUXILIARY LINES

Construct:

Ex. 1. An isosceles triangle, given the base and the radius of the circumscribed circle.

Ex. 2. A right triangle, given the radius of the circumscribed circle and one leg.

Ex. 3. A right triangle, given the radius of the circumscribed circle and an acute angle.

Ex. 4. A right triangle, given the radius of the inscribed circle and an acute angle.

[SUG. Draw the inscribed circle and at its center construct an angle equal to the supplement of the given angle.]

Ex. 5. A triangle, given one side, an adjacent angle, and the radius of the inscribed circle. (See Ex. 4.)

Ex. 6. A triangle, given one side, an adjacent angle, and the radius of the circumscribed circle.

271. Construction of circles.—The problem of constructing a required circle may often be reduced to another construction problem. For the construction of a circle frequently *reduces to the problem of finding a point (the center of the required circle) which answers given conditions.* (See § 268.) After the center has been found, it is usually easy to determine the radius.

Ex. Construct a circle which shall touch two given intersecting lines and have its center in another given line.

This problem is equivalent to the problem of finding a point which shall be in a given line and be equidistant from two other given lines. (See Ex. 3, p. 169.)

In some cases, however, the construction of a required circle must be made by an independent method.

EXERCISES: GROUP 47

Construction of Circles

Ex. 1. Construct a circle having a given radius, passing through a given point, and touching a given line.

Ex. 2. Construct a circle having a given radius, having its center in a given line, and touching another given line.

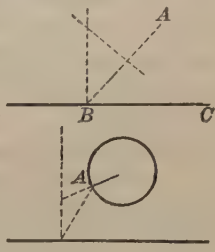
Ex. 3. Construct a circle having a given radius, and passing through two given points.

Ex. 4. Construct a circle which touches two given parallel lines and passes through a given point.

Ex. 5. Construct a circle which passes through two given points and has its center on a given line.

Ex. 6. Construct a circle which passes through a given point A and touches a given line BC at a given point B.

[Sug. Draw AB and at B construct a $\perp$ to BC.]



Ex. 7. Construct a circle which touches a given line and also touches a given circle at a given point A.

EXERCISES: GROUP 48

Problems Solved by Various Methods

Ex. 1. Through a given point, draw a line which shall cut two given intersecting lines so as to form an isosceles triangle.

Ex. 2. Construct an isosceles triangle, given the altitude and one leg.

Ex. 3. Draw a line which shall be terminated by the sides of a given angle, shall equal a given line, and be parallel to another given line.

Ex. 4. Construct a triangle, given one side, an adjacent angle, and the difference of the other two sides.

Ex. 5. Find a point in a given circle at a given distance from a given point.

Ex. 6. Construct a parallelogram, given a side, an angle, and a diagonal.

Ex. 7. Through a given point within an angle, draw a straight line terminated by the sides of the angle and bisected by the given point.

[Sug. Draw a line from the vertex of the angle to the given point and produce it its own length through the point.]

Ex. 8. Construct an isosceles triangle, given the base and the angle at the vertex.

Ex. 9. Draw a circle with given radius which shall touch a given circle at a given point.

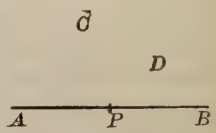
Ex. 10. Construct a right triangle, given the hypotenuse and the altitude upon the hypotenuse.

Ex. 11. Construct a triangle, given the altitude and the angles at the extremities of the base.

Ex. 12. Find the locus of the points of contact of tangents drawn from a given point to a series of circles having a given center. (Use §§ 212 and 244.)

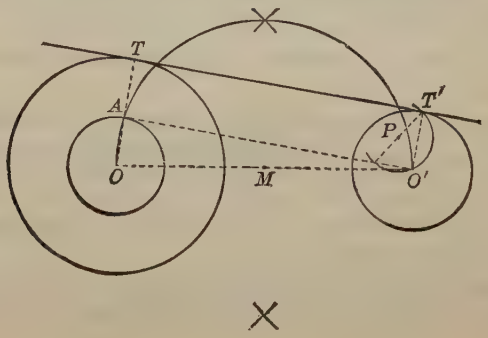
Ex. 13. Given a line AB and two points C and D on the same side of AB. Find a point P in AB, such that $\angle APC = \angle BPD$.

[Sug. Draw a $\perp$ from C to AB and produce it its own length, etc.]

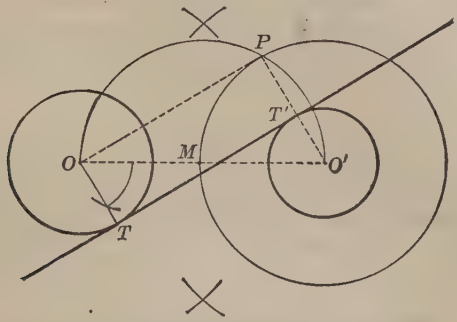


Ex. 14. Given a line AB and two points C and D on the same side of AB. Find a point P in AB, such that $CP+PD$ shall be a minimum.

Ex. 15. To construct a common external tangent to two given circles.



Ex. 16. To construct a common internal tangent to two given circles.



EXERCISES: GROUP 49

Practical Applications

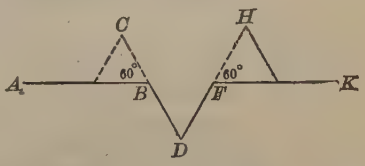
Ex. 1. Show how to find the center of a given circle by use of a carpenter's square.

Ex. 2. Show how to bisect a given angle by the use of a carpenter's square.

Ex. 3. By use of squared paper, divide a line $1\frac{1}{2}$ inches long into 5 equal parts. Into 7 equal parts. Into 3 equal parts. Can you make this division on paper ruled in only one direction?

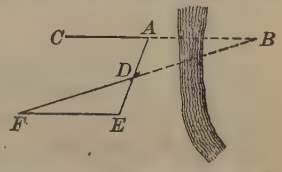
Ex. 4. Make up and work an example similar to Ex. 3.

Ex. 5. To extend a straight line AB beyond an obstacle, as a building, we may proceed as follows:



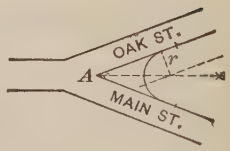
At B construct an angle $ABC = 60°$. Produce CB to D, etc. Let the student complete the construction and prove that his method is correct.

Ex. 6. To find the distance between two points A and B, one of which (B) is inaccessible, we may proceed as follows:

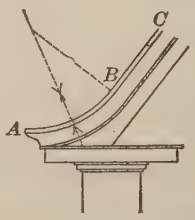


Extend BA to C. Measure a convenient line AD and extend AD to E, making $DE = AD$. From E run a line $\| AC$ and meeting BD extended at F. Measure EF. Prove that $EF = AB$.

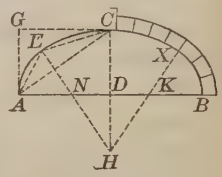
Ex. 7. If two streets meet, as in the diagram, show how a curve of given radius r may be made to take the place of the angle A and be tangent with the curb of the two streets.



Ex. 8. Draw an easement cornice (AB) tangent to the rake cornice BC, and passing through a required point A. (The same construction is used in laying out the easements of stair rails, etc.) (Use Ex. 6, p. 173.)

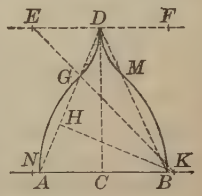


Ex. 9. A *segmental arch* is a compound curve composed of the arcs of three circles. The method of constructing a segmental arch is as follows: Let AB be the span and CD the altitude of the required arch. Complete the rectangle $GADC$. Draw the diagonal AC. Bisect the angles GAC and GCA. Let the bisectors meet at E. Draw EH perpendicular to AC, meeting AB at N and CD produced at H. Make DK equal to DN. Then show that N is the center and NA the radius, H the center and HE the radius, and K the center and KB the radius, for the arcs composing the arch. (See Hanstein's *Constructive Drawing*.)



[Sug. At E draw a line perpendicular to EH. Then $\angle NEA = \angle NAE$. (Complements of equal angles are equal), etc.]

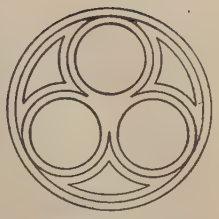
Ex. 10. What is called a *Persian arch* may be constructed as follows: Let AB be the span and CD the altitude of the required arch. Draw the isosceles triangle ADB. Divide AD into three equal parts at H and G. Construct $HK \perp AG$ at H, and meeting AB produced at K. Produce KG to meet EF which has been drawn through $D \parallel AB$. With K as a center and KA as a radius, and E as a center and EG as a radius, describe arcs meeting at G. Prove that these arcs have a common tangent at G, and therefore form a compound curve. (See Hanstein's *Constructive Drawing*.)



Ex. 11. Construct a Persian arch in which the arc DG = arc AG.

[Sug. Bisect line AD, instead of trisecting it.]

Ex. 12. Construct the adjoining ornamental design by first constructing the trefoil involved and locating the center of the equilateral triangle used in constructing the trefoil. (See Ex. 6, p. 124.)

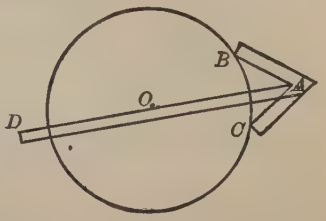


Ex. 13. The instrument $ABCD$ is used as a means of finding quickly the center of a circular disc or end of a cylindrical roller.

The edge AB = edge AC.

Also the edge AD bisects the angle BAC.

Prove that AD must pass through the center O when the points B and C are on the circle. (For a similar exercise, see Ex. 17, p. 137.)



In how many different positions must the instrument be applied to a circle in order to determine the center of the circle?

EXERCISES: GROUP 50

Review Exercises

Make a list of the properties of

Ex. 1. Arcs of a circle.

Ex. 2. Chords.

Ex. 3. Diameters.

Ex. 4. Central angles.

Ex. 5. Inscribed angles.

Ex. 6. Tangents.

BOOK THREE

PROPORTION; SIMILAR POLYGONS

THEORY OF PROPORTION

272. Ratio has been defined, and its use briefly indicated, in §§ 230, 231 (p. 138).

273. A **proportion** is a statement that two ratios are equal

Ex. 1. $\frac{12}{18}=\frac{2}{3}$. Thus, when a fraction is reduced to its lowest terms, a proportion is formed.

Ex. 2. $\frac{a}{b}=\frac{c}{d}$, or $a : b = c : d$.

The last example is read "the ratio of a to b equals the ratio of c to d," or "a is to b as c is to d."

274. The **terms of a proportion** are the four quantities used in the proportion. In a proportion, the **extremes** are the *first* and *last* terms; the **means** are the *second* and *third* terms.

275. A **fourth proportional** is the fourth term of a proportion in which the first three terms are three given numbers taken in order.

Thus, in $2 : 3 = 10 : x$, x is the fourth proportional to 2, 3, and 10.
In $a : b = c : d$, what is the fourth proportional?

276. A **mean proportional** is a term in a proportion used twice as a mean between two given numbers.

Thus, in $4 : 6 = 6 : 9$, 6 is the mean proportional between 4 and 9
In $a : b = b : c$, what is the mean proportional?

USEFUL PROPERTIES OF PROPORTIONS

277. Property I. *In any proportion, the product of the extremes is equal to the product of the means.*

Given the proportion $a : b = c : d$.

To prove $ad = bc$.

Proof. 1. $\frac{a}{b} = \frac{c}{d}$.	1. Hyp.
2. Multiplying each member by bd, $ad = bc$.	2. Ax. 4.
	Q.E.D.

278. Cor. *The mean proportional between two quantities is equal to the square root of their product.*

For, if $a : b = b : c$, then $b^2 = ac$ (§ 277);
or $b = \sqrt{ac}$. (Ax. 6.)

279. The **efficiency value** of §§ 277 and 278 is that these principles enable us to convert any given proportion into an equation, and often enable us to find the value of an unknown term in the proportion by solving the equation.

Ex. Find the value of x in the proportion $3 : 5 = x : 25$.
Using § 277, we obtain $5x = 3 \times 25$.
Hence $x = 15$. *Ans.*

Ex. 1. Construct a proportion in which the mean proportional is x, the last term is a, and the first term is 5.

Find the value of x

Ex. 2. In $4 : 8 = x : 20$.

Ex. 3. In $2 : 6 = 5 : x$.

Ex. 4. In $a : p = q : x$.

Ex. 5. In $x : 6 = .6 : 1.5$.

Ex. 6. In $.06 : 2x - 1 = 3 : 4$.

Ex. 7. In $x : 4b = 9b : 3b$.

Ex. 8. In $.05 : 3.2 = 1.5 : x$.

Ex. 9. Find the fourth proportional to 2, 3, and 6. Also to 3, $\frac{1}{2}$, $\frac{2}{3}$.

[Sug. Denote the required fourth proportional by x.]

Ex. 10. Find the mean proportional between 3 and 6. Also between .4 and .016.

Ex. 11. Find the mean proportional between $\frac{1}{2}$ and $\frac{2}{9}$.

Ex. 12. Find the mean proportional between $3p$ and $27p^3$.

Ex. 13. Find the mean proportional between $a+b$ and $a-b$.

Ex. 14. In the square $ABCD$, P is a point on AB, and Q on CD, such that $PB=CQ$. AQ and PD intersect at O. Prove $AO=OD$.

280. Property II. *If the product of two quantities is equal to the product of two other quantities, one pair may be made the extremes and the other pair the means of a proportion.*

Given $ad=bc$.

To prove $a:b=c:d$.

Proof. 1. $ad=bc$.	1. Hyp.
2. Dividing each member by bd, $\frac{a}{b}=\frac{c}{d}$	2. Ax. 5.
or, $a:b=c:d$.	**Q.E.D.**

281. Cor. 1. *If the first and third terms of a proportion are equal, the second and fourth terms are equal.*

Thus, if $a:x=a:y$, then $x=y$.
Let the student supply the proof.

282. Cor. 2. *If three terms of one proportion are equal to the corresponding three terms of another proportion, the fourth terms of the two proportions are equal.*

Thus, if $a:b=c:x$, and $a:b=c:y$, then $x=y$.
Let the student supply the proof.

Ex. 1. Write $ab=pq$ as a proportion in eight different ways.

Ex. 2. Write $x(x+1)=6$ as a proportion in which x is the first term. Write $x^2=15$ as a proportion in which 3 is the first term.

Ex. 3. Write $3x=4y$ as a proportion in which x and y are the first two terms.

Find the value of $x : y$ if

Ex. 4. $5x=7y$ (see Ex. 3).

Ex. 5. $ax=by$.

Ex. 6. $ax+bx=py+qy$.

Ex. 7. $ax-by=cx-dy$.

283. Property III. *If four quantities are in proportion, they are in proportion by* **alternation;** *that is, the first term is to the third as the second is to the fourth.*

Given the proportion $a : b=c : d$.

To prove $a : c=b : d$.

Proof. 1.	$a : b=c : d$.	1. Hyp.
2.	$\therefore ad=bc$.	2. § 277.
3.	$\therefore a : c=b : d$.	3. § 280. Q.E.D.

284. Property IV. *If four quantities are in proportion, they are in proportion by* **inversion;** *that is, the second term is to the first as the fourth is to the third.*

Given the proportion $a : b=c : d$.

To prove $b : a=d : c$.

Proof. Let the student supply the proof.

Ex. 1. Construct a triangle ABC no two of whose sides are equal. Also on the sides AB and BC draw the equilateral triangles AFB and BGC outside the triangle ABC. Draw FC and AG. Prove $\angle FBC=\angle ABG$. Also prove $FC=AG$.

Ex. 2. Draw a circle O and in it the diameters AB and FH perpendicular to each other. Draw AH and OR, the altitude from O to AH. Prove $OR=\frac{1}{2}AR$.

Ex. 3. Draw a line AB, 2 in. long. Also draw the locus of the vertices of the right angles in all right triangles having AB as their hypotenuse. Write a description of the locus.

Ex. 4. Transform $x : a = b : c$ so that a shall be the third term.

Ex. 5. Transform the proportion of Ex. 4 so that c shall be the third term.

Ex. 6. So that x shall be the last term.

Ex. 7. Write $x = \frac{ab}{c}$ as a proportion in which x is the last term. Do the same with $x = \frac{ab}{2c}$.

Ex. 8. Write $x^2 = 3$ as a proportion in which x is a mean proportional. Do the same with $x = \sqrt{a^2 - b^2}$.

285. Property V. *If four quantities are in proportion, they are in proportion by* **addition;** *that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the last term.*

Given the proportion $a : b = c : d$.

To prove $a + b : b = c + d : d$.

Proof.

1. $\frac{a}{b} = \frac{c}{d}$. — 1. Hyp.

2. Adding one to each member of the equality, — 2. Ax. 2.

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

That is, $a + b : b = c + d : d$. **Q.E.D.**

Let the student show also that $a + b : a = c + d : c$.

By use of the property of addition, transform

Ex. 1. $12 : 3 = 8 : 2$.

Ex. 2. $2x - 5 : 5 = 3x - 7 : 7$.

Ex. 3. $\sqrt{x+3} - 5 : 5 = \sqrt{2x-1} - 7 : 7$.

Ex. 4. What is the efficiency value of the property of addition when applied to certain proportions?

Ex. 5. Make up and solve an example similar to Ex. 2. To Ex. 3.

286. Property VI. *In a series of equal ratios, the sum of all the first terms is to the sum of all the second terms as the first term in any of the given ratios is to its second term.*

Given $a : b = c : d = e : f$.

To prove $a+c+e : b+d+f = a : b$.

Proof. 1. $ab = ab$.	1. Ident.
2. $a : b = c : d \quad \therefore bc = ad$.	2. Hyp. § 277.
3. $a : b = e : f \quad \therefore be = af$.	3. Hyp. § 277.
4. Adding, $b(a+c+e) = a(b+d+f)$.	4. Ax. 2.
5. Hence $a+c+e : b+d+f = a : b$.	5. § 280.
	Q.E.D.

By use of § 286, transform

Ex. 1. $4 : 1 = 12 : 3 = 8 : 2$.

Ex. 2. $\dfrac{x+1}{y+3} = \dfrac{p+3}{q-2} = \dfrac{2-p}{5-q}$.

Ex. 3. Find the value of $\dfrac{x}{y}$ if $\dfrac{x-a-2}{y-b-5} = \dfrac{a}{b} = \dfrac{2}{5}$.

287. Property VII. *The products of the corresponding terms of two or more proportions are in proportion.*

Given $a : b = c : d$, $e : f = g : h$, and $j : k = l : m$.

To prove $aej : bfk = cgl : dhm$.

Proof. 1. $\dfrac{a}{b} = \dfrac{c}{d}, \dfrac{e}{f} = \dfrac{g}{h}, \dfrac{j}{k} = \dfrac{l}{m}$.	1. Hyp.
2. $\dfrac{aej}{bfk} = \dfrac{cgl}{dhm}$,	2. Ax. 4.
or $aej : bfk = cgl : dhm$.	Q.E.D.

Ex. 1. If $x : y = 10 : 5$ and $p : q = 3 : 7$, find the value of $px : qy$.

Ex. 2. If $p : q = 1 : 5$ and $p : \frac{1}{q} = 4 : 5$, find the value of p.

288. Property VIII. *Like powers or like roots of the terms of a proportion are in proportion.*

Given the proportion $a : b = c : d$.

To prove $a^n : b^n = c^n : d^n$, and $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Proof.

1. $\frac{a}{b} = \frac{c}{d}$. — 1. Hyp.

2. Raising both members to the n^{th} power, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$, or $a^n : b^n = c^n : d^n$. — 2. Ax. 6.

3. In like manner, $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$. — 3. Ax. 6.

Q.E.D.

Ex. 1. If $x : y = 2 : 3$, find the value of $x^3 : y^3$.

Ex. 2. If $\sqrt[3]{a} : \sqrt[3]{b} = 2 : 5$, find the value of $a : b$.

Ex. 3. If $\sqrt{x} : 2 = \sqrt{y} : 7$, find the value of $x : y$.

Ex. 4. If $\sqrt{x} : \sqrt{x+1} = 3 : 5$, find the value of x.

Ex. 5. If $\sqrt{y} : \sqrt{6} = \sqrt{y+5} : \sqrt{10}$, find the value of y.

Ex. 6. State some of the efficiency values in the principle of § 288.

289. General principle.—Sometimes the two terms in a ratio have a common unit of measure (see § 231); sometimes they do not. Thus, in the ratio $\frac{1 \text{ ft. } 5 \text{ in.}}{1 \text{ yd.}}$, a unit of measure common to both terms is 1 inch. But the terms of the ratio $\frac{5 \text{ in.}}{\sqrt{2} \text{ in.}}$ have no common unit of measure.

However, by the method of limits it may be shown that *if two ratios, in each of which the terms have a common unit of measure, are equal, under like circumstances two ratios whose terms have no common unit of measure* (that is, are incommensurable) *are equal.* For instance, see § 235 (p. 140).

Ex. 1. Given $bx : y = bp : q$. Prove $x : y = p : q$.

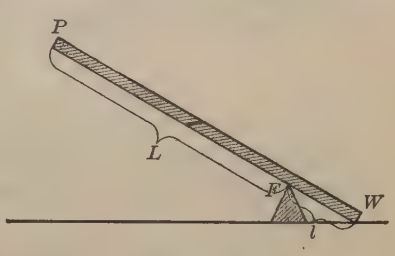
Ex. 2. PW is a lever and F a fulcrum. The lengths of the arms of the lever are denoted by L and l.

W = weight to be lifted (expressed in pounds).

P = no. lb. of force applied at P, which will raise the weight W.

Then $P \times L = l \times W$.

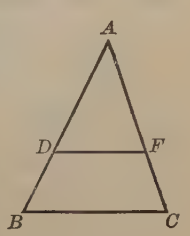
Express this formula as a proportion.



Ex. 3. In Ex. 2, if $W = 200$ lb., $l = 1$ ft., $L = 5$ ft., find P.

Ex. 4. What is the efficiency value of the law of the lever stated in Ex. 2?

290. Lines divided proportionally.—If a given line divides two sides of a triangle so that the corresponding segments are in proportion, the sides are said to be divided proportionally. Thus, on the adjoining figure, the sides AB and AC of the triangle ABC are divided proportionally by the line DF if $AD : DB = AF : FC$.



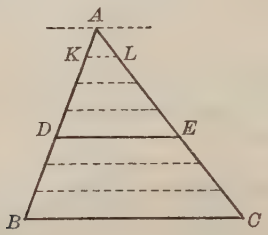
Ex. 1. Two tangents to a circle form an angle of 70°. Find the number of degrees in each of the two intercepted arcs.

Ex. 2. On a circle find a point equidistant from two given radii.

Ex. 3. Construct the locus of the vertices of all triangles having a given base of $1\frac{1}{2}$ in. and a vertex angle of 30°.

Proposition I. Theorem

291. *A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.*



Given the triangle ABC and the line $DE \parallel$ base BC and intersecting the sides AB and AC in the points D and E, respectively.

To prove $\frac{AD}{DB}=\frac{AE}{EC}$.

Proof. 1. Take any common unit of measure of AD and DB, as AK, and let it be contained in AD m times and in DB n times. Then $\frac{AD}{DB}=\frac{m(AK)}{n(AK)}=\frac{m}{n}$.	1. § 232.
2. Through A and the points of division of AD and DB, draw lines $\parallel BC$ and meeting AC.	2. § 100.
3. These lines will divide AE into m parts, and EC into n parts, all equal.	3. § 168.
4. Hence $\frac{AE}{EC}=\frac{m(AL)}{n(AL)}=\frac{m}{n}$.	4. Why?
5. Hence $\frac{AD}{DB}=\frac{AE}{EC}$.	5. Ax. 1. Q.E.D.

292. Cor. 1. By addition (§ 285),

$$AD+DB : AD = AE+EC : AE.$$

That is, $AB : AD = AC : AE.$

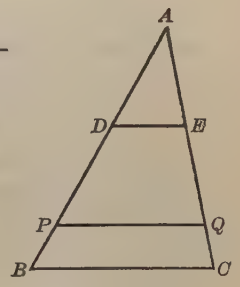
In like manner $AB : DB = AC : EC.$

Or in general language, *if a line parallel to the base cuts the sides of a triangle, a side is to a segment of that side as the other side is to the corresponding segment of the second side.*

In this case also, the sides AB and AC are said to be divided proportionally.

As a matter of convenience in making references, the principle stated in § 291 will be regarded as covering also the properties stated in § 292.

293. Cor. 2. On the adjoining diagram,



if $\frac{PB}{AP} = \frac{QC}{AQ}$, $\frac{PB}{QC} = \frac{AP}{AQ}$ (§ 283);

also, if $\frac{DP}{AP} = \frac{EQ}{AQ}$, $\frac{DP}{EQ} = \frac{AP}{AQ}$ (§ 283).

$\therefore \frac{PB}{QC} = \frac{DP}{EQ}$ (Ax. 1).

Hence, *the segments cut off on two transversals by a series of parallels are proportional.*

If the diagram of Prop. I were so constructed that

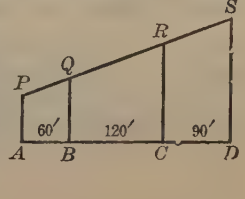
Ex. 1. $AD = 12$, $DB = 8$, and $AC = 15$, find AE and EC.

Ex. 2. $AE = 2DB$, $AD = 5$, and $EC = 10$, find DB.

Ex. 3. $AD = a$, $DB = b$, $EC = c$, find AE.

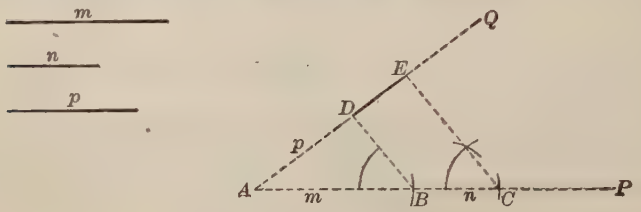
Ex. 4. If, on the diagram for § 293, $AD = 5\frac{3}{5}$, $DP = 6$, $PB = 2\frac{2}{5}$, $AC = 12$, find AE, EQ, QC.

Ex. 5. In the adjoining figure, PA, QB, RC, SD are $\perp AD$. AD is the frontage on one street and PS the frontage on another street, $PS = 360$ ft., $AB = 60$ ft., $BC = 120$ ft., $CD = 90$ ft. Find PQ, QR, and RS, the frontages of the lots on the other street.



Proposition II. Problem

294. *To construct the fourth proportional to three given line-segments.*



Given the line-segments m, n, p.

To construct a fourth proportional to m, n, and p.

Construction. 1. Draw two lines AP and AQ, making a convenient angle A.	1. Post. 1.
2. On AP mark off $AB=m$, $BC=n$; on AQ, $AD=p$.	2. Post. 2.
3. Draw DB.	3. Post. 1.
4. Through C draw $CE \parallel DB$ and meeting AQ at E.	4. § 100.
Then DE is the fourth proportional required.	
Proof. 1. $AB : BC = AD : DE$.	1. § 291.
2. Hence $m : n = p : DE$.	2. Ax. 9.
	Q.E.F.

Ex. 1. Construct a fourth proportional to three lines, $\frac{3}{4}$ in., 1 in., and $1\frac{1}{2}$ in. long.

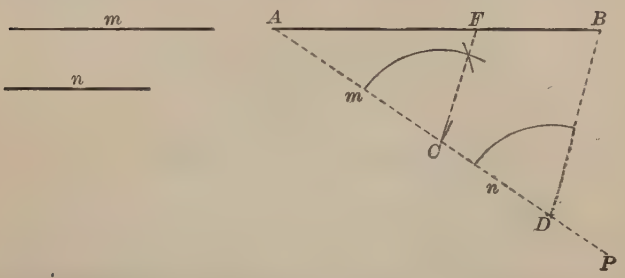
Ex. 2. Given 2 in. : $1\frac{1}{2}$ in. $= 1\frac{1}{2}$ in. : x, construct the line x.

Ex. 3. Given any two lines a and b, and $a : b = b : y$, construct the line y.

Ex. 4. Given any three lines a, b, c, and given $x = \frac{ab}{c}$, construct the line x.

Proposition III. Problem

295. *To divide a given line-segment into parts proportional to two given line-segments.*



Given the line-segments AB, m, and n.

To divide AB into parts proportional to m and n.

Construction. 1. Draw the line AP, making any convenient angle with AB.	1. Post. 1.
2. On AP mark off $AC=m$, and $CD=n$. Draw DB.	2. Post. 2.
3. Through C draw $CF \parallel DB$ and meeting AB at F.	3. § 100.
Then AF and FB are the required parts of AB.	
Proof. 1. $AF : FB = m : n$.	1. Why?

Q.E.F.

Ex. 1. Draw a line 2 in. long and divide it into two parts which shall be in the ratio of 5 to 3.

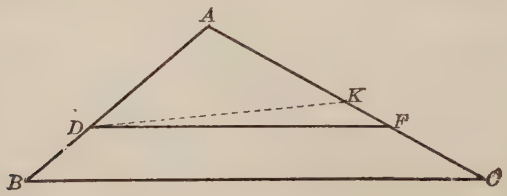
Ex. 2. Draw any two lines and denote them by a and b. Then draw a line $1\frac{1}{2}$ in. long and divide it into parts which shall be in the ratio of a to b.

Ex. 3. Divide a given straight line into parts which shall be proportional to three other given straight lines, denoted by m, n, p.

Ex. 4. Draw a line of convenient length and divide it into parts proportional to 2, 3, and 4.

Proposition IV. Theorem

296. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given the $\triangle ABC$ and the line DF intersecting AB and AC so that $AB : AD = AC : AF$.

To prove $DF \parallel BC$.

Proof. 1. Through D draw the line $DK \parallel BC$ and meeting AC at K.	1. § 100.
2. Then $AB : AD = AC : AK$.	2. § 292.
3. But $AB : AD = AC : AF$.	3. Hyp.
4. $\therefore AK = AF$.	4. § 282.
5. Hence the point K falls on F and the line DK coincides with the line DF.	5. § 57.
6. But then $DK \parallel BC$.	6. Constr.
7. $\therefore DF \parallel BC$.	7. DF coincides with DK, which $\parallel BC$. **Q.E.D.**

If the diagram of Prop. IV were so constructed that

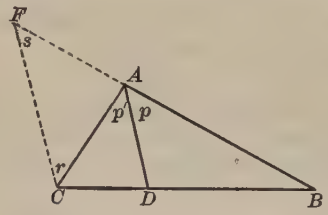
Ex. 1. $AD = 12$, $DB = 4$, $AF = 15$, and $FC = 5$, would $DF \parallel BC$?

Ex. 2. $AB = 20$, $AD = 16$, $AC = 25$, and $AF = 18$, would $DF \parallel BC$?

Ex. 3. $AD = 3DB$, and $AF = 3FC$, would $DF \parallel BC$?

Proposition V. Theorem

297. *The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Given the $\triangle ABC$, with the line AD bisecting the $\angle BAC$ and meeting BC at D.

To prove $CD : DB = AC : AB$.

Proof. 1. Draw the line $CF \parallel AD$, and meeting BA produced at F.	1. § 100.
2. Then $CD : DB = FA : AB$.	2. § 291.
3. But $\angle r = \angle p'$.	3. § 101.
4. $\angle s = \angle p$.	4. § 102.
5. Also $\angle p' = \angle p$.	5. Hyp.
6. $\therefore \angle r = \angle s$.	6. Why?
7. $\therefore \triangle ACF$ is isosceles, and $AC = AF$.	7. Why?
8. $\therefore CD : DB = AC : AB$.	8. Ax. 9.
	Q.E.D.

If the diagram of Prop. V were so constructed that

Ex. 1. $AC = 5$, $AB = 10$, and $CD = 4$, find DB.

Ex. 2. $AC = 2.4$, $AB = 5.6$, and $DB = 3.5$, find CD.

Ex. 3. $AB = 16$, $AC = 12$, and $BC = 14$, find CD and DB.

[Sug. Let $DC = x$, $DB = 14 - x$, etc.]

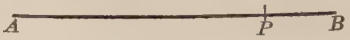
Ex. 4. $AB = 22$, $AC = 11$, and $CB = 21$, find CD and DB.

[Sug. Solve orally by separating 21 into two parts which are as 1 : 2.]

Ex. 5. Solve similarly, if $AC = 7$, $AB = 21$, and $CB = 20$.

Ex. 6. Also, if $AC = 6$, $AB = 9$, and $CB = 10$.

298. A **line divided internally** is a line divided into two parts by a point taken between the extremities of the given line.



Thus the line AB is divided internally at the point P into the segments PA and PB.

To divide a given line internally in a given ratio (as 2 : 7), divide the given line into a number of equal parts, equal to the *sum* of the terms of the ratio (as 2+7, or 9, equal parts).

A **line divided externally** is a line whose parts are considered to be the segments included between a point on the line produced and the extremities of the given line.

Thus, if the line AB is produced through A to P', the line AB is divided externally at the point P' into the two segments $P'A$ and $P'B$.

To divide a given line externally in a given ratio (as 2 : 7), divide the given line into a number of equal parts, equal to the *difference* of the terms of the ratio (as 7−2, or 5 equal parts) and produce the line till the produced part equals the smaller term of the ratio times the unit line found.

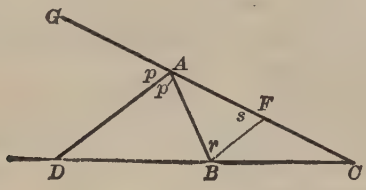
Ex. 1. Draw a line $2\frac{1}{4}$ in. long and divide it both internally and externally into two parts as 5 : 2.

Ex. 2. Draw a line $1\frac{3}{4}$ in. long and divide it externally into two parts as 2 : 5.

Ex. 3. By use of ruler and compasses construct a rectangle 2 in long and 1 in. wide. Draw the locus of a point which moves outside the rectangle so as always to be $\frac{3}{4}$ in. from it. Write a description of this locus.

Proposition VI. Theorem

299. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*



Given the $\triangle ABC$ with its exterior $\angle BAG$ bisected by the line AD meeting the opposite side produced at D.

To prove $DB : DC = AB : AC$.

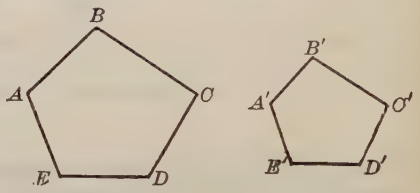
Proof. 1. Draw the line $BF \parallel AD$ and meeting AC at F.	1. § 100.
2. Then $DB : DC = AF : AC$.	2. Why?
3. But $\angle r = \angle p'$.	3. Why?
4. And $\angle s = \angle p$.	4. Why?
5. Also $\angle p' = \angle p$.	5. Why?
6. $\therefore \angle r = \angle s$.	6. Why?
7. $\therefore \triangle ABF$ is isosceles, and $AB = AF$.	7. Why?
8. Substituting AB for its equal, AF, $DB : DC = AB : AC$.	8. Ax. 9.
	Q.E.D.

Note.—If in the above diagram $AB = AC$, AD will be parallel to BC, and the theorem is not true unless we regard two parallel lines as meeting at an infinite distance.

SIMILAR POLYGONS

300. Similar polygons are polygons having their corresponding angles equal and their corresponding sides proportional.

Thus, if the figures $ABCDE$ and $A'B'C'D'E'$ are similar, the angles A, B, C, etc., must equal the angles A', B', C', etc., respectively; also $AB : A'B' = BC : B'C' = CD : C'D'$, etc.



Hence it is constantly to be borne in mind that similarity in shape or form of rectilinear figures involves two distinct properties:

1. *The corresponding angles are equal.*
2. *The corresponding sides are proportional.*

It should also be clearly realized that one of these properties may be true of two figures, and not the other.

Thus, in the rectangle A and the parallelogram B, the corresponding sides are proportional but the corresponding angles are not equal.

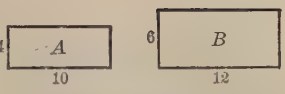
Also, in the rectangle C and the square D, the corresponding angles are equal but the corresponding sides are not proportional.

However, it will be found that, in the case of *triangles*, if one of the two properties is true, the other must be true also.

The **ratio of similitude** in two similar figures is the ratio of any two corresponding sides in those figures.

Ex. 1. Are any two squares similar?

Ex. 2. A and B are rectangles. Are they similar?

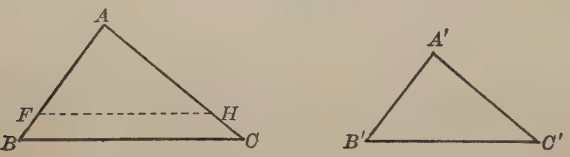


Ex. 3. If, in the similar pentagons of § 300, $AB=30$, $BC=47$, $CD=36$, $ED=33$, $AE=24$, and $A'B'=21$, find the other sides of the second pentagon.

Ex. 4. Make up a problem similar to Ex. 2, and answer it.

Proposition VII. Theorem

301. *If two triangles have the angles of one respectively equal to the angles of the other, the triangles are similar.*



Given the $\triangle ABC$ and $A'B'C'$ with $\angle A=\angle A'$, $\angle B=\angle B'$, and $\angle C=\angle C'$.

To prove the $\triangle ABC$ and $A'B'C'$ similar.

Proof. 1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ shall coincide with its equal, $\angle A$, and $B'C'$ shall take the position FH.	1. Geom. Ax. 2.
2. Then $\angle AFH=\angle B$.	2. Hyp.
3. $\therefore FH \parallel BC$.	3. Why?
4. $\therefore AB : AF=AC : AH$.	4. § 291.
5. Hence $AB : A'B'=AC : A'C'$.	5. Ax. 9.
6. In like manner, by placing $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle B'$ shall coincide with its equal $\angle B$, it may be proved that $AB : A'B'=BC : B'C'$.	6. Reasons 1–5 above.
7. Hence $\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.	7. Ax. 1.
8. Hence $\triangle ABC$ and $A'B'C'$ are similar.	8. § 300. Q.E.D.

302. Cor. 1. *If two triangles have two angles of one respectively equal to two angles of the other, the triangles are similar.*

303. Cor. 2. *If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.*

304. Cor. 3. *If two triangles are each similar to the same triangle, they are similar to each other.*

305. Cor. 4. *A line parallel to the base of a triangle cuts off a triangle similar to the given triangle.*

306. The **efficiency value** of the principles arrived at in §§ 301–304 is that by them the labor of proving two triangles similar is reduced to that of proving two angles of one triangle equal to two angles of the other; or, in the case of two right triangles, of proving only one pair of angles equal.

It is often convenient to use the **symbol** $\sim$ as a substitute for the words " is similar to."

Ex. 1. Are two rhombuses ever similar? Are two parallelograms always similar?

Ex. 2. Construct two similar rectangles in which the ratio of similitude is 2 : 3.

Ex. 3. *ABCD* is a quadrilateral inscribed in a circle. The sides *AB* and *DC* are produced to meet at *E*, and the chords *AC* and *BD* are drawn. Prove that the triangles *ACE* and *BDE* are similar.

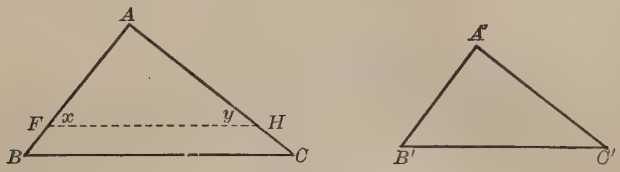
Can you point out another pair of similar triangles on the figure?

Ex. 4. The three sides of a given triangle are 8, 10, and 12 inches, respectively. If a line 9 inches long, parallel to the longest side, is terminated by the other two sides, find the segments into which it divides them.

Ex. 5. Write out a list of the theorems used in proving the theorem that, if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Proposition VIII. Theorem

307. *If the corresponding sides of two triangles are proportional, the triangles are similar.*



Given the $\triangle ABC$ and $A'B'C'$, in which $AB : A'B' = AC : A'C' = BC : B'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof. 1.	On AB mark off AF equal to $A'B'$, and on AC take $AH = A'C'$. Draw FH.	1. Post. 2.
2. Then	$AB : AF = AC : AH$.	2. Hyp.
3.	$\therefore FH \parallel BC$.	3. § 296.
4.	$\therefore \triangle AFH \sim \triangle ABC$.	4. § 305.
5. Hence	$AB : AF = BC : FH$.	5. § 300
6. But	$AB : A'B' = BC : B'C'$.	6. Hyp.
7. But	$AF = A'B'$.	7. Const.
8.	$\therefore FH = B'C'$.	8. § 282.
9. Hence	$\triangle AFH \cong \triangle A'B'C'$.	9. § 87.
10.	$\therefore \triangle ABC \sim \triangle A'B'C'$.	10. Ax. 9.
		Q.E.D.

Ex. 1. In $\triangle ABC$, $AB = 9$, $AC = 12$, $BC = 15$; also in $\triangle A'B'C'$, $A'B' = 6$, $A'C' = 9$, $B'C' = 10$. Are the two triangles similar?

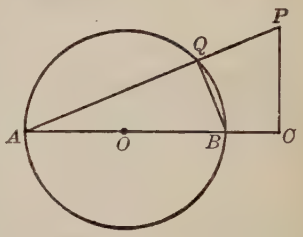
Ex. 2. In proving Prop. VIII, why cannot $\triangle A'B'C'$ be applied to $\triangle ABC$ as in the proof of Prop. VII?

Ex. 3. The bases of two isosceles $\triangle$ are 6 in. and 4 in., and their legs are 12 in. and 8 in., respectively. Are the $\triangle$ similar?

Ex. 4. Make up an example similar to Ex. 1. To Ex. 3.

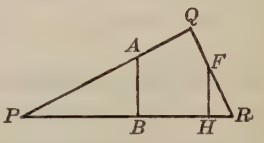
EXERCISES: GROUP 51

Ex. 1. Given AB a diameter of the circle O; AB produced to C and $PC \perp AC$; AP a line intersecting the circle in Q. Prove $\triangle APC \sim \triangle AQB$.



Ex. 2. Draw a triangle and a line parallel to one of its sides and meeting the other two sides produced. Prove that the two triangles thus formed are similar.

Ex. 3. Given the triangle PQR, Q a right angle; AB and $FH \perp PR$. Prove that the diagram contains three similar triangles.



Ex. 4. AB is the hypotenuse of the right triangle ABC. At A and B perpendiculars to AB are erected to meet BC and AC produced at P and Q, respectively. Prove $\triangle PAC \sim \triangle BCQ$.

Ex. 5. $ABCD$ is a parallelogram. P is any point on the side CD; AP produced meets the side BC produced at Q. Prove that the figure contains three similar triangles.

Ex. 6. In a given circle draw two chords AB and CD intersecting at F. Draw lines joining the extremities of the chords. How many pairs of similar triangles does the figure contain? Give proof.

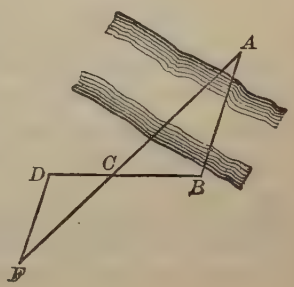
Ex. 7. From a given point outside a circle, two secants are drawn to the circle. Lines are drawn connecting the points where the secants cut the circle. Prove that two pairs of similar triangles are thus formed.

Ex. 8. To determine the distance AB across a stream, construct any convenient line BD, and $DF \parallel BA$. Determine the point C where FA crosses DB.

Then prove $\triangle FDC \sim \triangle CAB$.

If $CB = 240$ yd., $DC = 160$ yd., and $DF = 248$ yd., compute AB.

State the efficiency value of similar triangles which is illustrated by the preceding problem.

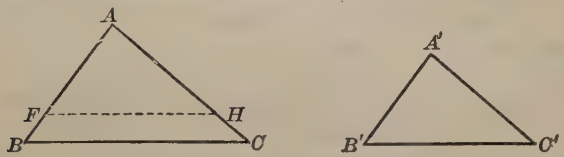


Ex. 9. Extend the non-parallel sides of a trapezoid until they meet. Then prove that the two triangles thus formed are similar.

Ex. 10. From a point A on a circle, a diameter AD is drawn and also two unequal chords, AB and AC, on opposite sides of AD. The tangent at D meets AB produced at E and AC produced at F. Prove $\angle F = \angle ABC$, and $\triangle ABC$ and AEF similar.

Proposition IX. Theorem

308. *If two triangles have an angle of one equal to an angle of the other, and the sides including these angles proportional, the triangles are similar.*



Given the $\triangle ABC$ and $A'B'C'$, in which $\angle A = \angle A'$ and $AB : A'B' = AC : A'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

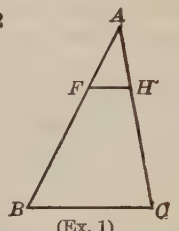
Proof. 1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ shall coincide with its equal, $\angle A$, and $B'C'$ take the position FH.	1. Geom. Ax. 2.
2. Then $AB : AF = AC : AH$.	2. Hyp.
3. Hence $FH \parallel BC$.	3. Why?
4. $\therefore \triangle ABC \sim \triangle AFH$.	4. Why?
5. $\therefore \triangle ABC \sim \triangle A'B'C'$.	5. Why?
	Q.E.D.

309. Cor. *Two isosceles triangles in which the vertex angles are equal are similar.*

Ex. 1. In the $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$; in the similar $\triangle A'B'C'$, $A'B' = 4.2$. Find $A'C'$ and $B'C'$.

Ex. 2. The triangle ABC has a right angle at B, and D is a point on AC such that AB is a mean proportional between AC and AD. Prove $\triangle ABD \sim \triangle ABC$. Also prove $BD \perp AC$.

EXERCISES: GROUP 52

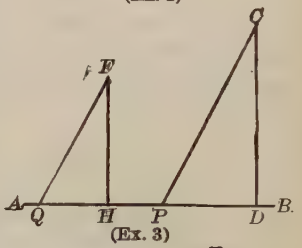


(Ex. 1)

Ex. 1. Given $\triangle ABC$ with $AF = \frac{1}{3}AB$ and $AH = \frac{1}{3}AC$. **Prove** $\triangle AFH \sim \triangle ABC$.

[Sug. If $AF = \frac{1}{3}AB$, then $\frac{AF}{AB} = \frac{1}{3}$ (Ax. 5), etc.]

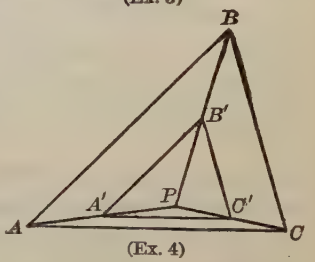
Ex. 2. A and A' are corresponding vertices in a pair of similar triangles. Prove that medians drawn from A and A' divide the triangles into two pairs of similar triangles.



(Ex. 3)

Ex. 3. Given FH and $CD \perp$ straight line AB; $FQ \parallel CP$. **Prove** $\triangle QFH \sim \triangle CPD$.

Ex. 4. Given P any point within the $\triangle ABC$; A', B', C', midpoints of PA, PB, and PC, respectively. **Prove** $\triangle A'B'C' \sim \triangle ABC$.

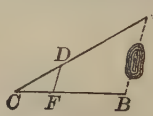


(Ex. 4)

Ex. 5. State and prove the theorem similar to that of Ex. 4, when P is a point outside of the triangle ABC.

Ex. 6. Prove the theorem of Ex. 4 when P is a point on one of the sides of $\triangle ABC$.

Ex. 7. A certain tree casts a shadow 132 ft. long. At the same time a rod 10 ft. high casts a shadow 12 ft. long. Compute the height of the tree. (See Ex. 3.)



(Ex. 8)

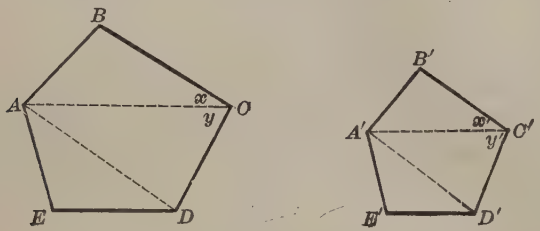
Ex. 8. To find the distance AB, a convenient position C is taken and CA and CB are drawn. On CA, CD is measured off $= \frac{1}{5}CA$, and on CB, CF is measured off $= \frac{1}{5}CB$. Prove $\triangle CDF \sim \triangle CAB$. (See Ex. 1.) If DF is measured and found to be 217 ft., how long is AB?

Ex. 9. In a triangle ABC, the bisectors of the angles B and C meet the sides opposite these angles in Q and R, respectively. If AQ, QC, AR, and RB are 9, 15, 8, and 10 inches, respectively, find BC.

Ex. 10. If one of two similar triangles is isosceles, prove that the other is isosceles also.

Proposition X. Theorem

310. *If two polygons are similar, they can be divided into triangles which are similar and similarly placed.*



Given the similar polygons $ABCDE$ and $A'B'C'D'E'$, divided into triangles by the diagonals AC, AD, and $A'C'$, $A'D'$, drawn from the corresponding vertices A and A'.

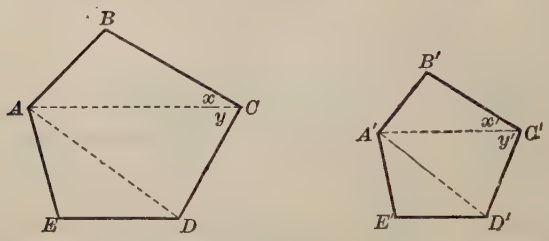
To prove that $\triangle ABC$, ACD, ADE are similar to the $\triangle A'B'C'$, $A'C'D'$, $A'D'E'$, respectively.

Proof. 1. $\angle B = \angle B'$.	1. § 300.
2. Also $AB : A'B' = BC : B'C'$.	2. § 300.
3. $\therefore \triangle ABC \sim \triangle A'B'C'$.	3. § 308.
4. Again $\angle BCD = \angle B'C'D'$.	4. § 300.
5. Also $\angle x = \angle x'$.	5. § 300.
6. Subtracting, $\angle y = \angle y'$.	6. Why?
7. But $BC : B'C' = CD : C'D'$.	7. § 300.
8. And $BC : B'C' = AC : A'C'$.	8. § 300.
9. Hence $AC : A'C' = CD : C'D'$.	9. Ax. 1.
10. $\therefore \triangle ACD \sim \triangle A'C'D'$.	10. § 308.
11. In like manner, $\triangle ADE \sim \triangle A'D'E'$.	11. Reasons 1–3 above. **Q.E.D.**

Ex. Construct two rectangles whose bases are 8 and 10, and altitudes 4 and 5, respectively. Show that the rectangles are similar according to § 300. Draw a pair of corresponding diagonals and show by § 308 that the corresponding $\triangle$ formed are similar.

PROPOSITION XI. THEOREM (CONVERSE OF PROP. X)

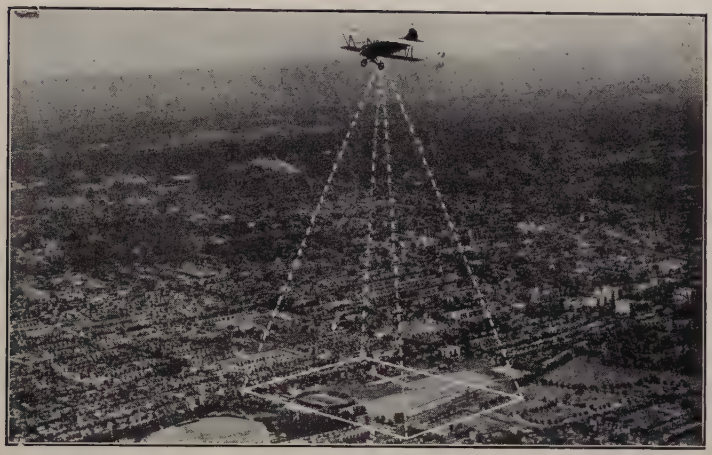
311. *If two polygons can be divided into triangles which are similar and similarly placed, the polygons are similar.*



Given the two polygons $ABCDE$ and $A'B'C'D'E'$, in which the $\triangle ABC$, ACD, ADE are similar, respectively, to the $\triangle A'B'C'$, $A'C'D'$, $A'D'E'$, and are similarly placed.

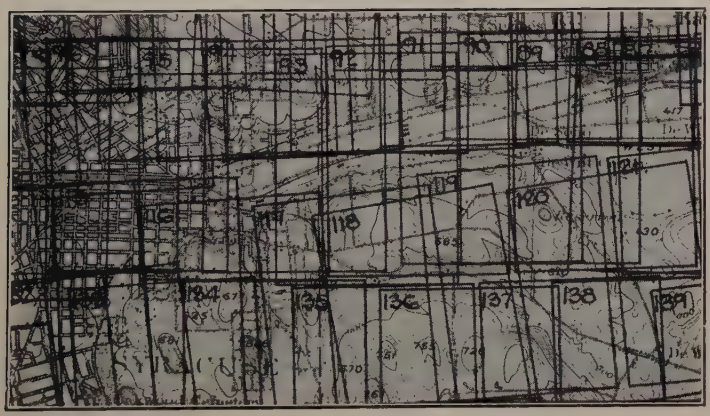
To prove $ABCDE \sim A'B'C'D'E'$.

Proof. 1. $\angle B = \angle B'$.	1. Corr. $\measuredangle$ of similar $\triangle$ are $=$. (§ 300.)
2. Also $\angle x = \angle x'$.	2. Why?
3. And $\angle y = \angle y'$.	3. Why?
4. Adding, $\angle BCD = \angle B'C'D'$.	4. Why?
5. Likewise, $\angle CDE = \angle C'D'E'$, $\angle BAE = \angle B'A'E'$, etc.	5. Reasons 1–4.
6. Also $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.	6. Corr. sides of similar $\triangle$ are proportional. (§ 300.)
7. $\frac{BC}{B'C'} = \frac{AC}{A'C'}$, and $\frac{CD}{C'D'} = \frac{AC}{A'C'}$.	7. Why?
8. Hence $\frac{BC}{B'C'} = \frac{CD}{C'D'}$.	8. Why?
9. So $\frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{AE}{A'E'}$.	9. Reasons 6–8.
10. $\therefore ABCDE \sim A'B'C'D'E'$.	10. § 300. Q.E.D.



AERIAL MAPPING

The airplane above is taking photographs of the country to be mapped. The approximate area covered by each exposure of the film is outlined in white. The exposures are made every few seconds so that the photographs overlap each other as shown on the index map below. What geometric principles are illustrated in taking the pictures? What two things are affected by the height of the plane? From the index map, how can the photographer tell whether or not he has covered his territory properly? (*Airmap Corporation of America*)



AN AERIAL MAP

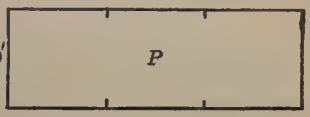
Here is a section of a completed aerial map. What advantages has it over the old-type maps? Can you suggest some uses to which a city or county might put such a map? Why would it be just as accurate if enlarged to cover the wall of a room? (*Airmap Corporation of America*)

312. Abbreviated form of statement.—It is often convenient to write a series of equal ratios, like those used in the preceding proof, as follows:

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\left(\frac{AC}{A'C'}\right)=\frac{CD}{C'D'}=\left(\frac{AD}{A'D'}\right)=\frac{DE}{D'E'}=\frac{AE}{A'E'}.$$

In this case we inclose in parenthesis a ratio that is used merely to show the equality of two other ratios.

313. Drawing to scale.—A rectangular building lot is measured and found to be 50 ft. wide and 150 ft. deep. The diagram P represents this lot, a line 1 in. long on the diagram standing for a line 100 ft. long on the lot. This method of drawing is called drawing to scale.

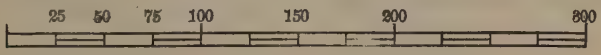


Drawing to scale is making a drawing of an object so that every line on the drawing is a given fractional part (or multiple) of the corresponding line on the object.

Thus, if the scale of the drawing is $\frac{1}{10}$, each line on the drawing equals $\frac{1}{10}$ of the corresponding line on the object.

Maps and architects' drawings of buildings are familiar examples of drawings to scale.

The scale of a drawing should always be indicated on the drawing. The following shows a convenient method of doing this:



Scale, 1 Inch to 100 Miles

$$\left(\text{or Scale}=\frac{1}{6{,}336{,}000}\right).$$

Ex. 1. Construct a drawing scale 3 inches long, similar to the one just given, but to the scale of 1 inch to 20 feet. Of 1 inch to 1 yard. Of 1 inch to 40 miles.

Ex. 2. Using a scale of $\frac{1}{12}$, draw a line to represent 18 in. To represent 30 in. To represent 2 yd.

Ex. 3. Using a scale of $\frac{1}{12}$, construct a drawing to represent a rectangle 30 in. long and 15 in. wide.

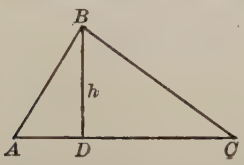
Ex. 4. The adjoining diagram represents the plan of a cellar. The scale of the diagram is $\frac{1}{240}$. Find the perimeter of the cellar in feet.

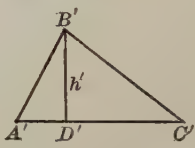
Ex. 5. State some of the advantages of being able to make drawings to scale.

Ex. 6. Using a convenient scale, make a drawing of the school grounds. On this drawing represent the main school building.

Proposition XII. Theorem

314. *The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.*





Given the similar $\triangle ABC$, $A'B'C'$, and h, h', any two corresponding altitudes in these $\triangle$.

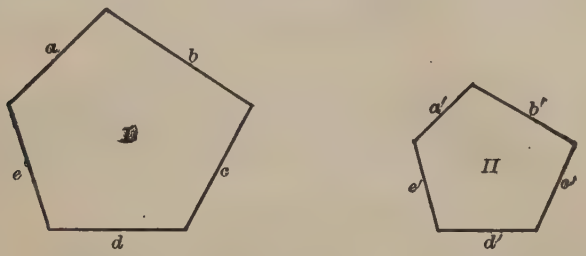
To prove $\frac{h}{h'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}$.

Proof. 1.	In the rt. $\triangle ABD$ and $A'B'D'$, $\angle A=\angle A'$.	1. Why?
2.	$\therefore \triangle ABD \sim \triangle A'B'D'$.	2. Why?
3.	$\therefore \frac{h}{h'}=\frac{AB}{A'B'}$.	3. Why?
4. But	$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}$.	4. Why?
5. Hence	$\frac{h}{h'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}$.	5. Why?

Q.E.D.

Proposition XIII. Theorem.

315. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Given the similar polygons I and II with the sides a, b, c, etc., of I corresponding to the sides a', b', c', etc., of II; the perimeter of I being denoted by P, and that of II by P'.

To prove $P : P' = a : a'$.

Proof. 1. $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}$.	1. Why?
2. Hence $\frac{a+b+c+d+e}{a'+b'+c'+d'+e'}=\frac{a}{a'}$,	2. § 286.
or $\frac{P}{P'}=\frac{a}{a'}$.	**Q.E.D.**

316. Cor. *In two similar polygons, any two corresponding lines are to each other as any other two corresponding lines; and the perimeters are to each other as any two corresponding lines.*

Ex. 1. If the perimeter of a given field is 210 rods, and a side of this field is to a corresponding side of a similar field as 3 : 2, find the perimeter of the second field.

Ex. 2. One rectangular farm is twice as large, in length and width, as another similar farm. How many times longer is the fencing inclosing the first farm than that inclosing the second?

EXERCISES: GROUP 53

Ex. 1. Prove that any two given squares are similar to each other.

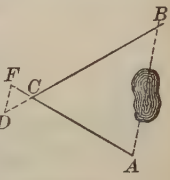
Ex. 2. On a given map, two places A and B are represented as $2\frac{3}{4}$ inches apart. How many miles is it from A to B if the scale of the map is 50 miles to the inch?

Ex. 3. How many miles apart are A and B (Ex. 2) if the scale of the map is $\frac{1}{1{,}000{,}000}$?

Ex. 4. Prove in full (without using § 316) that the perimeters of two similar polygons are to each other as any two corresponding diagonals.

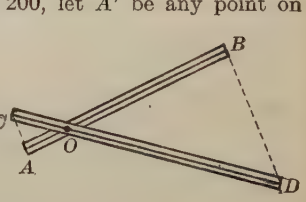
Ex. 5. To find the distance AB, take a convenient point C and measure AC and CB. Produce AC to F, making $CF=\frac{1}{5}AC$. Also produce BC to D, making $CD=\frac{1}{5}CB$. Prove $\triangle CDF\sim\triangle ACB$.

If DF is measured and found to be 136 yd., how long is AB?



Ex. 6. On the diagram of Ex. 4, p. 200, let A' be any point on PA, $A'B' \parallel AB$, and $B'C' \parallel BC$. Prove $A'C' \parallel AC$.

Ex. 7. AB and CD are two rods fastened together at O, and each able to rotate about O. If $BO=\frac{1}{3}AO$ and $DO=\frac{1}{3}CO$, prove that in every position of the rods $BD=\frac{1}{3}AC$.



Ex. 8. If, on the diagram of Prop. XII, $AC=18$, $A'C'=12$, and $BD=15$, find $B'D'$.

Ex. 9. Two sides of a triangle are 8 and 12, and the altitude on the third side is 6. In a similar triangle, the side corresponding to 8 is 10. In the second triangle, compute one of the other two sides and the altitude corresponding to the altitude 6.

Ex. 10. In the triangle ABC, AM bisects the angle BAC, CM is perpendicular to AM and when produced meets AB in K. D is the midpoint of BC. Prove $DM \parallel BK$. If $AB=7$, $BC=5.5$, and $AC=4.6$, find the length of BK and of DM.

Ex. 11. Using a convenient scale, make a drawing to represent the outline of the schoolroom floor. On this drawing can you represent the teacher's desk? The other desks in the room?

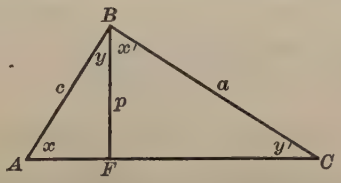
Proposition XIV. Theorem

317. *In any right triangle,*

I. *The perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two triangles, each similar to the given triangle;*

II. *The perpendicular is the mean proportional between the segments of the hypotenuse;*

III. *Each leg of the given right triangle is the mean proportional between the hypotenuse and the segment adjacent to that leg.*



Given the right $\triangle ABC$ and BF the perpendicular from B to the hypotenuse AC.

To prove I. $\triangle ABF \sim \triangle BFC \sim \triangle ABC$.

II. $AF : p = p : FC$.

III. $AC : c = c : AF$, and $AC : a = a : FC$.

Proof. I. 1. In the rt. $\triangle ABF$ and ABC, $\angle x = \angle x$.	1. Ident.
2. $\therefore \triangle ABF \sim \triangle ABC$.	2. § 303.
3. In the rt. $\triangle BFC$ and ABC, $\angle y' = \angle y$.	3. Why?
4. $\therefore \triangle BFC \sim \triangle ABC$.	4. Why?
5. $\therefore \triangle ABF \sim \triangle BFC \sim \triangle ABC$.	5. § 304.
II. 1. In the similar $\triangle ABF$ and BFC, $AF : p = p : FC$.	1. § 300.
III. 1. In the similar $\triangle ABC$ and ABF, $AC : c = c : AF$.	1. Why?
2. In the similar $\triangle ABC$ and BFC, $AC : a = a : FC$.	2. Why? **Q.E.D.**

318. Cor. 1. *The perpendicular to the diameter from any point on a circle is a mean proportional between the segments of the diameter; and the chord joining the point to an extremity of the diameter is a mean proportional between the diameter and the segment of the diameter adjacent to the chord.*

319. Cor. 2. On the diagram of Prop. XIV, $p^2 = AF \times FC$; $c^2 = AF \times AC$; and $a^2 = FC \times AC$.

On the diagram of Prop. XIV,

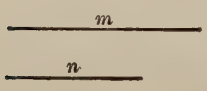
Ex. 1. If $AF = 4$ and $FC = 9$, find BF. Find AB and BC.

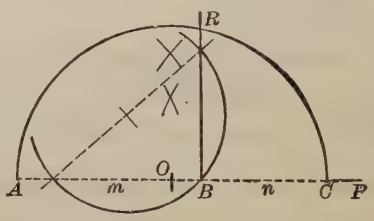
Ex. 2. If $BC = 1.2$ and $AC = 1.6$, find FC, AF, and BF.

Ex. 3. If $AF = .08$ and $BF = .16$, find BC.

Proposition XV. Problem

320. *To construct the mean proportional between two given line-segments.*





Given the line-segments m and n.

To construct the mean proportional between m and n.

Construction. 1. Draw a line AP and on it mark off $AB = m$ and $BC = n$.	1. Posts. 1 and 2.
2. Bisect AC at O.	2. § 130
3. With O as a center and AO as a radius, describe the semicircle ARC.	3. Post. 3.
4. At B erect $BR \perp AC$, meeting ARC at R. BR is the mean proportional required.	4. § 252.
Proof. Let the student supply the proof.	Q.E.F.

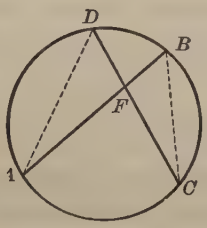
Ex. 1. Construct the mean proportional between two lines 1 in. and 2 in. long, respectively. Also between lines of $\frac{3}{4}$ in. and $1\frac{1}{4}$ in.

Ex. 2. Taking any line as 1, construct a line equal to $\sqrt{2}$. Also $\sqrt{6}$.

Ex. 3. Construct a line $\sqrt{5}$ inches long.

Proposition XVI. Theorem

321. *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given the circle $ADBC$, with the chords AB and CD intersecting at the point F.

To prove $AF \times FB = CF \times FD$.

Proof. 1. Draw AD and CB.	1. Post. 1.
2. Then in the $\triangle AFD$ and CFB, $\angle A = \angle C$.	2. § 243.
3. Also $\angle D = \angle B$.	3. Why?
4. $\therefore$ $\triangle AFD \sim \triangle CFB$.	4. Why?
5. $\therefore AF : CF = FD : FB$.	5. Why?
6. $\therefore AF \times FB = CF \times FD$.	6. Why? **Q.E.D.**

322. Cor. *The product of the segments of a chord that passes through a fixed point within a circle is constant in whatever direction the chord is drawn.*

If the diagram of Prop. XVI were so constructed that

Ex. 1. $AF=12$, $FB=8$, and $FC=16$, find DF.

Ex. 2. $AF=2FB$, $FC=9$, and $DF=8$, find AF and FB.

Ex. 3. $FC=6$, $DF=4$, and $AB=11$, find AF and FB.
[Sug. Let $AF=x$, etc.]

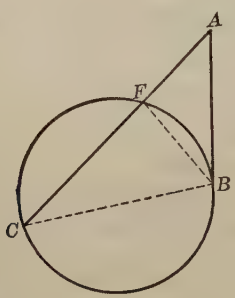
Ex. 4. $AF=a$, $FB=b$, and $DF=c$, find FC.

Ex. 5. $FC=p$, $DF=q$, and $AB=r$, find AF.
How do you interpret the $\pm$ sign in your answer?

Ex. 6. In the acute triangle ABC, let the altitudes BD and CE intersect in the point O. Prove that $BO : OC = OE : OD$.

Proposition XVII. Theorem

323. *If, from a point outside a circle, a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.*



Given AB, a tangent, and AC, a secant, to the circle BCF, and AF, the external segment of the secant.

To prove $AC : AB = AB : AF$.

Proof. Let the student supply the proof.

324. Cor. *If, from a given point without a circle, a secant is drawn, the product of the secant and its external segment is constant, in whatever direction the secant is drawn.*

For the product of each secant thus drawn and its external segment equals the square of the tangent, which is constant.

If the diagram of Prop. XVII were so constructed that

Ex. 1. $AC=9$ and $AF=4$, find AB.

Ex. 2. $AC=4\ AF$ and $AB=12$, find AC.

Ex. 3. $AC=.64$ and $AB=.32$, find AF.

Ex. 4. O is the center, radius $=15$, and $AO=21$, find $AC\times AF$.

Ex. 5. The diameter AB of a circle is 15 in., P is a point in AB $4\frac{1}{2}$ in. from the center. CD is the chord through P perpendicular to AB. Find the length of CD. Also of the chords CB and AC.

Ex. 6. In a circle whose radius is 18 in., P is a point 14 in. from the center. Through P a chord 24 in. long is drawn. Find the segments of this chord made by the point P.

Ex. 7. A chord 5 in. long is prolonged to a point P, from which a tangent 6 in. long is drawn to the circle. How far is the chord prolonged?

Ex. 8. A and B are two points on a railway curve which is an arc of a circle. If the length of the chord AB is 200 ft. and the shortest distance from the midpoint of the curve AB to the chord is 4 ft., find the radius of the arc.

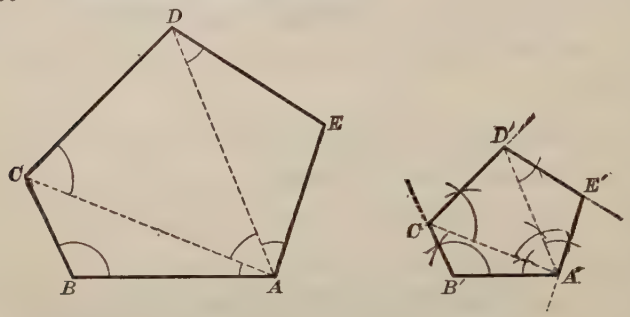
Ex. 9. How far from a circle whose radius is 6 must a point be taken in order that the whole secant from that point through the center shall be twice the tangent from that point?

Ex. 10. ABC is a right triangle with the right angle at B. BF is the altitude on AC. If $BC=10$, and $AF=8$, find AC to two decimal places.

Ex. 11. The sides of a triangle are 10, 12, and 15 in. In a triangle similar to it, the shortest side is 16 in. Find the length of the other two sides.

Proposition XVIII. Problem

325. *Upon a given line-segment as side, to construct a polygon similar to a given polygon.*



Given the polygon $ABCDE$ and the line $A'B'$.

To construct a polygon similar to $ABCDE$ on $A'B'$ as a side corresponding to AB.

Construction and proof. 1. In polygon $ABCDE$, from any convenient vertex, as A, draw diagonals.	1. Post. 1.
2. On $A'B'$ construct $\angle A'B'C' = \angle ABC$ and $\angle B'A'C' = \angle BAC$, letting $B'C'$ and $A'C'$ meet at C'.	2. § 90.
3. Then $\triangle A'B'C' \sim \triangle ABC$.	3. Why?
4. In like manner, on $A'C'$ construct $\triangle A'C'D'$ similar to $\triangle ACD$, and so on.	4. Reasons 2 and 3.
5. Then $A'B'C'D'E' \sim ABCDE$, and $A'B'C'D'E'$ is the polygon required.	5. § 311. **Q.E.F.**

Ex. 1. In the triangle ABC, K is any point on AB, and P, Q, T are the midpoints respectively of AC, AK, and KC. Prove that $KQPT$ is a parallelogram.

Ex. 2. Two unequal circles are tangent to each other externally The radius of the larger circle is 12, and the common external tangent meets the line of centers produced at an angle of 30°. Find the radius of the smaller circle.

Ex. 3. Construct a triangle whose sides are $2''$, $1\frac{3}{4}''$, $1\frac{1}{2}''$. Construct a similar triangle whose shortest side is $1''$.

Ex. 4. The base of a triangle is 20 ft. The other two sides of the triangle are 16 ft. and 10 ft. A line parallel to this base cuts 2 ft. from the lower end of the shorter side. Find the segments of the other side and the length of the line parallel to the base.

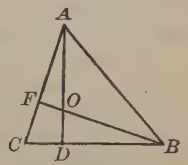
EXERCISES: GROUP 54

Similar Triangles

Let the student make a list of all the conditions that make two triangles similar. (See §§ 301, 302, 303, etc.)

Ex. 1. Given $AD \perp BC$, and $BF \perp AC$. **Prove** $\triangle ADC$ and BFC similar.

Ex. 2. In the figure of Ex. 1, prove the $\triangle AOF$ and BFC similar. What other triangle on this figure is similar to $\triangle BFC$?

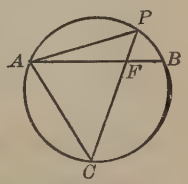


(Ex. 1)

Ex. 3. Two isosceles triangles are similar if a base angle of one equals a base angle of the other.

Ex. 4. Prove that the diagonals and bases of a trapezoid together form a pair of similar triangles.

Ex. 5. Given $\overset{\frown}{AC} = \overset{\frown}{BC}$. **Prove** $\triangle APC$ and AFC similar.



(Ex. 5)

Ex. 6. AB is the diameter of a circle, BD is a tangent, and AD intersects the circle at E. Prove the triangles ABE and ADB similar.

Ex. 7. BC is a chord in a circle, AQ is the diameter perpendicular to BC and meeting it at N; AP is any chord intersecting BC in M. Prove the $\triangle AMN$ and APQ similar.

Ex. 8. The triangle ABC is inscribed in a circle; the bisector of the angle A meets BC in D and the circle in P. Prove the triangles BAD and APC similar.

Ex. 9. Two circles intersect in the points A and B. AC and AD are each a tangent in one circle and a chord in the other. Prove the $\triangle ABC$ and ABD similar.

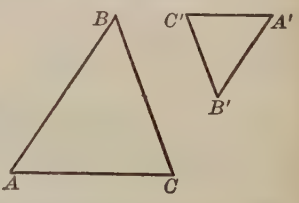
[Sug. Prove $\angle BAD = \angle ACB$, etc.]

Ex. 10. Prove that two rectangles are similar if two adjacent sides of one are proportional to the corresponding sides of the other.

Ex. 11. In a triangle ABC, $BC=9$ in. and $CA=15$ in. On BC, P is a point such that $CP=5$ in., and on CA, Q is a point such that $QC=3$ in. Prove $\triangle CPQ \sim \triangle ABC$.

Ex. 12. $ABCD$ is a quadrilateral inscribed in a circle. AB and DC produced meet at P. If the diagonals AC and DB are drawn, how many pairs of similar triangles are on the figure?

Ex. 13. If two triangles have the sides of one respectively parallel to the sides of the other, the triangles are similar.



[Sug. Call pairs of corresponding angles A and A', B and B', C and C', respectively.

(1) To have one of these pairs, as $\angle A$ and $\angle A'$, supplementary, and the other two pairs, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$, equal is impossible. (§ 113.)

(2) To have two or three pairs supplementary is also impossible, for (§ 108) the sum of the $\measuredangle$ of two $\triangle$ cannot exceed 360°.

(3) Hence $\angle A=\angle A'$, $\angle B=\angle B'$, $\angle C=\angle C'$ (§ 118), and the triangles are similar. (§ 301.)]

Ex. 14. If two triangles have the sides of one triangle respectively perpendicular to the sides of the other triangle, the triangles are similar.

326. Proof that lines are proportional.—In order to prove that certain lines are proportional, or have proportional relations, it is usually best to *show that the given lines are corresponding sides of similar triangles.*

In order to find the required pair of triangles, it is a help to mark the four lines (which are to form the required proportion) in some special way, as by brackets or colored crayon or pencil, and then to select the two triangles which contain these four lines as sides.

Sometimes, however, other methods of proof are used (as the theorems of §§ 321 and 323); but these, if investigated, are usually found to be the method of similar triangles in disguise.

EXERCISES: GROUP 55

PROPORTIONAL LINES

Ex. 1. On the figure of Ex. 1, p. 213, prove $AD \times BC = DF \times AC$, and $BC \times OD = BO \times FC$.

Ex. 2. On the figure of Ex. 5, p. 213, prove $CP : CA = CA : CF$. (Hence, as P moves, the product of what two lines is constant?)

Ex. 3. The diagonals of a trapezoid divide each other into proportional segments.

Ex. 4. In the isosceles $\triangle ABC$, $AB = AC$; on the side AB the point P is taken so that PC equals the base. Prove $AB \times PB = \overline{BC}^2$.

Ex. 5. In a triangle the median to the base bisects all lines parallel to the base and terminated by the sides.

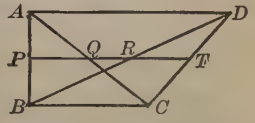
Ex. 6. If PQ is any line through F, a fixed point in the line AB, and AP and BQ are perpendicular to PQ, show that the ratio $PF : FQ$ is constant.

Ex. 7. The triangle ABC is inscribed in a circle. F is the midpoint of the arc AC, and BF intersects the line AC in E. Prove $AB : BC = AE : EC$. (Use § 297.)

Ex. 8. If two circles intersect, the common chord, if produced, bisects the common tangent. (Use § 323.)

Ex. 9. If two circles intersect, tangents drawn to the two circles from any point in the common chord produced are equal.

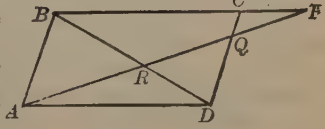
Ex. 10. Given AD, PT, and BC $\|$; prove $PQ = RT$.



[SUG. Show that $\frac{PQ}{BC} = \frac{RT}{BC}$, by showing them equal to a common ratio.]

Ex. 11. Lines are drawn from a point O within the triangle to the vertices of the triangle ABC. From B', any point in OB, $B'A'$ is drawn parallel to BA and meeting OA in A', and $B'C'$ is drawn parallel to BC and meeting OC in C'. Prove $A'B' : AB = B'C' : BC$, and the triangles ABC and $A'B'C'$ similar.

Ex. 12. Given $ABCD$ a $\square$, and P any point in BC produced. Prove $\overline{AR}^2 = RQ \times RP$.



[SUG. Compare the similar $\triangle ABR$ and RQD; also the similar $\triangle ARD$ and RBP.]

EXERCISES: GROUP 56

Auxiliary Lines

Ex. 1. Prove that lines joining the midpoints of the sides of a quadrilateral taken in order form a parallelogram.

[Sug. Draw the diagonals of the quadrilateral and use § 171.]

Ex. 2. Lines joining the midpoints of the sides of a rectangle in order form a rhombus.

Ex. 3. Lines joining the midpoints of the sides of a rhombus in order form a rectangle.

Ex. 4. The common tangent of two circles divides the line of centers into segments which have the same ratio as the diameters of the circles.

[Sug. Draw radii to the points of contact.]

Ex. 5. AB and AC are the legs of an isosceles triangle, and BF is an altitude. Prove that $2AC \times FC = \overline{BC}^2$.

Ex. 6. ABC is an inscribed isosceles triangle of which AB and AC are the legs. AD is a chord meeting BC in E. Prove that $\overline{AB}^2 = AD \times AE$.

Ex. 7. Two circles touch at the point T. PTP' and QTQ' are lines drawn meeting the circles in P, Q and P', Q', respectively. Prove the triangles PTQ and $P'TQ'$ similar.

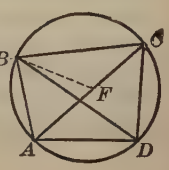
[Sug. Draw the common tangent at T.]

Ex. 8. Two circles touch at the point T, and through T three lines are drawn meeting the circles in P, Q, R, and P', Q', R', respectively. Prove the triangles PQR and $P'Q'R'$ similar.

Ex. 9. If A is the midpoint of CD, an arc of a circle, and AP is any chord intersecting the chord CD in E, prove that $AP \times AQ$ is a constant.

Ex. 10. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.

[Sug. Draw BF so that $\angle CBF = \angle ABD$, and use similar triangles.]



EXERCISES: GROUP 57

Theorems Proved by Various Methods

Ex. 1. In the figure on p. 207, show that $AB \times BF = BC \times AF$.

Ex. 2. In the same figure, if $FC = 3AF$, show that $\overline{AB}^2 : \overline{BC}^2 = 1 : 3$.

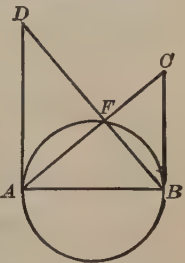
Ex. 3. AB is the diameter of a circle and PB is a tangent. If AP meets the circle in the point Q, prove that $AP \times AQ = \overline{AB}^2$.

Ex. 4. In similar triangles, corresponding medians have the same ratio as corresponding sides.

Ex. 5. A diameter AB is produced to the point C; CP is perpendicular to AC; PB produced meets the circle at Q. Prove the triangles AQB and PCB similar.

Ex. 6. If PA and PB are unequal chords in a circle, and CD is a line parallel to the tangent at P and crossing PA and PB at C and D, the triangles PAB and PCD are similar.

Ex. 7. **Given** AB a diameter and AD and BC tangents, AC and DB intersecting at a point F on the circle. **Prove** AB a mean proportional between the lines AD and BC.



(Ex. 7)

Ex. 8. ABC is a triangle with a right angle at B. P and S are points on AC, Q is a point on AB, and R on BC, such that $PQRS$ is a square. Prove that a side of this square is a mean proportional between AP and SC.

Ex. 9. A given angle has its vertex at O. P and P' are points on one side of this angle, and Q and Q' are points on the other side, the line-segment QQ' being much smaller than PP'. If $OP \times OP' = OQ \times OQ'$, prove $\triangle OPQ \sim \triangle OP'Q'$.

Ex. 10. AB and CD are both perpendicular to the line AOD, and $\angle BOA = \angle COD$. Write a correct proportion whose terms are BA, AO, OD, and CD.

Ex. 11. ABF is a right triangle with $AF \perp BF$. $ABCD$ is a rectangle such that DC intersects AF at Q and BF at P. If $AF = 8$ ft., $FB = 12$ ft., and $BC = 8$ in., find PC and DQ in inches. How many triangles on the figure are similar to each other?

Ex. 12. In the triangle ABC, $AB = BC = 25$ in., and $AC = 30$ in. Find the length of the parts of BC made by the altitude from A.

Ex. 13. A line drawn through the intersection of the diagonals of a trapezoid parallel to the bases and terminated by the legs is bisected by the diagonals.

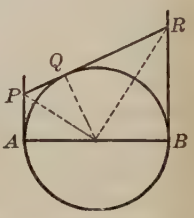
[Sug. See Ex. 10, p. 215.]

Ex. 14. If a chord is bisected by another chord, each segment of the first chord is a mean proportional between the segments of the second chord.

Ex. 15. If two circles are tangent externally, and a line is drawn through the point of contact and terminated by the circles, the chords intercepted in the two circles are to each other as the radii.

Ex. 16. Find the locus of the midpoints of lines in a triangle parallel to the base and terminated by the sides.

Ex. 17. Given AB the diameter, AP, PQR, BR, tangents. **Prove** $PQ \times QR$ a constant (=radius squared).



(Ex. 17)

EXERCISES: GROUP 58

Construction Problems

Given three lines a, b, c,

Ex. 1. Construct $x=\frac{ab}{c}$; also $x=\frac{ab}{2c}$; also $x=\frac{2ac}{b}$.

Ex. 2. Construct $x=\frac{b^2}{a}$; also $x=\frac{a^2}{b}$.

Ex. 3. Construct $x=\sqrt{ab}$; also $2x=\sqrt{ab}$.

Ex. 4. Construct $x=\sqrt{a^2-b^2}$, i.e., $\sqrt{(a+b)(a-b)}$.

Ex. 5. Construct $x=\sqrt{3ab}$, i.e., $\sqrt{(3a)b}$; also $x=\sqrt{2ab}$.

Ex. 6. Given a line denoted by 1, construct $\sqrt{3}$; also $\frac{1}{2}\sqrt{5}$.

Ex. 7. Given a line-segment a, construct another line-segment of length $a\sqrt{2}$; also $\frac{a}{2}\sqrt{2}$.

Ex. 8. Divide a line into three parts proportional to 2, $\frac{1}{2}$, $\frac{2}{3}$.

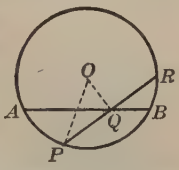
Ex. 9. Divide one side of a triangle into segments proportional to the other two sides.

Ex. 10. Divide a line into segments in the ratio $1 : \sqrt{2}$.

Ex. 11. Given a point P in the side AB of a triangle ABC; draw a line from P to AC produced so that the line drawn may be bisected by BC.

[Sug. Suppose the required line, PQR, drawn meeting BC in Q and AC in R. From P draw $PL \parallel AC$ and meeting BC in L. Compare the $\triangle PLQ$ and QRC.]

Ex. 12. Through a given point P in the arc determined by the chord AB, draw a chord which shall be bisected by AB.



[Sug. Suppose the required chord drawn, viz., PQR. Join the center O with P and Q. What kind of angle is OQP, etc.?]

Ex. 13. In an obtuse triangle draw a line from the vertex of the obtuse angle to the opposite side which shall be a mean proportional between the segments of the opposite side.

[Sug. Circumscribe a circle about the triangle and reduce the problem to the conditions of Ex. 12.]

Ex. 14. Find a point P in the arc subtended by the chord AB such that chord PA : chord $PB = 2 : 3$.

[Sug. Suppose the required construction made, and also the chord AB divided in the ratio $2 : 3$ at the point Q. How do the angles APQ and QPB compare?]

Ex. 15. Given the perimeter, construct a triangle similar to a given triangle.

Ex. 16. Given the altitude of a triangle, construct a triangle similar to a given triangle.

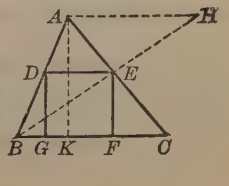
Ex. 17. In a given circle inscribe a triangle similar to a given triangle.

Ex. 18. About a given circle circumscribe a triangle similar to a given triangle.

Ex. 19. By drawing a line parallel to one of the sides of a given rectangle, divide the rectangle into two similar rectangles.

Ex. 20. Construct a circle which shall pass through two given points and be tangent to a given straight line.

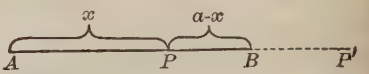
Ex. 21. Inscribe a square in a given triangle.



[Sug. If ABC is the given triangle, suppose $DGFE$ the required inscribed square. Join BE and produce it to meet $AH \parallel BC$. Prove $AH = AK$, etc.]

327. Algebraic analysis of problems.—*The conditions of a problem may often be stated as an algebraic equation. By solving the equation, the length of a desired line in terms of known lines may be obtained, and the problem may be solved by constructing the algebraic expression thus obtained.*

Ex. Find a point P in the line AB, such that $\overline{AP}^2=3\overline{BP}^2$.



Analysis and Construction. Denote AB by a, AP by x, and PB by $a-x$. Then $x^2=3(a-x)^2$.

$\therefore 2x^2-6ax=-3a^2$, and $x=\dfrac{3a\pm a\sqrt{3}}{2}$.

Construct $a\sqrt{3}$; whence construct $\dfrac{3a-a\sqrt{3}}{2}$. Lay off the line obtained, as AP, on AB. This gives the point P of internal division. Similarly, the construction of $\dfrac{3a+a\sqrt{3}}{2}$ gives P', the point of external division.

EXERCISES: GROUP 59

Problems Solved by Algebraic Analysis

Ex. 1. Find a point P in a given line AB, such that $\overline{AP}^2=2\overline{BP}^2$.

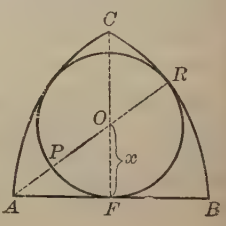
Ex. 2. Inscribe a square in a given semicircle.

Ex. 3. From a given line cut off a part which shall be a mean proportional between the remainder of the line and another given line.

Ex. 4. From a given point draw a secant to a circle so that the external segment shall equal half the secant.

Ex. 5. To inscribe a circle in a given equilateral gothic arch ABC. (See Ex. 6, p. 121.)

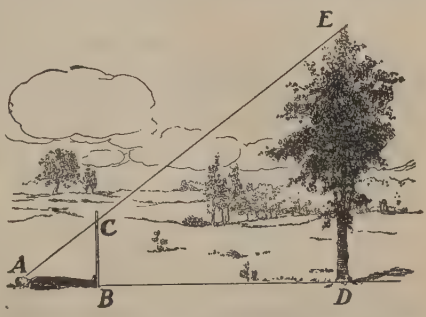
[Sug. Denote AB by a, the radius of the required circle by x, and then determine x in terms of a. Thus $AR=AB=a$, $AF=\frac{1}{2}a$, $AP=a-2x$. Then use § 323.]



EXERCISES: GROUP 60

Practical Applications

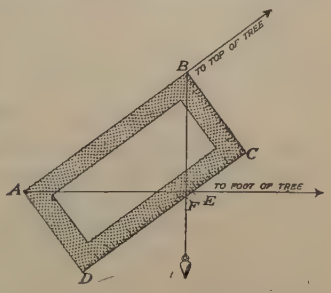
Ex. 1. In case the sun is not shining and shadows cannot be used, the height of an object, such as a tree or a steeple, can often be determined by a method indicated in the drawing. What distances must be measured, and why, to determine the height of the tree?



Ex. 2. Show how to find the height of a tree by placing a mirror in a horizontal position on the ground and standing so as to see the reflection of the top of the tree in the mirror. If the observer's eye is $5\frac{1}{2}$ ft. from the ground, the observer stands 6 ft. from the mirror, and the mirror is 120 ft. from the tree, how high is the tree?

Ex. 3. Foresters often determine the height of a tree by an instrument called Faustman's Height Measurer. The principle on which this instrument is constructed is shown in the diagram.

If the distance from A to the foot of the tree is 150 ft., $BC = 6$ inches, and $CF = 4\frac{1}{2}$ inches, find the height of the tree.

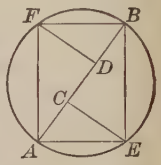


Ex. 4. The distance from A to B in Ex. 5, p. 206, might have been determined by a graphical method as follows: Measure AC, CB, and angle ACB. On paper make a drawing of the triangle ACB to a convenient scale. On this drawing measure the line which represents AB and hence determine the length of AB.

By use of this method, we are saved the labor of marking out and measuring the lines CD, CF, and DF.

Apply this method to the measurement of two objects in your neighborhood which are separated by an impassable barrier. In like

manner, show how the distance between two objects, only one of which is accessible, may be determined.



Ex. 5. Show how to find, by use of the graphical method, the distance between two places both of which are inaccessible.

Ex. 6 The strongest beam which can be cut from a given round log is found as follows: Take AB, a diameter of the log, and trisect it at C and D. Draw CE and $DF \perp AB$ and meeting the circle at E and F respectively. Draw AF, FB, BE, and AE.

Prove $AFBE$ a rectangle. Also that $FB : AF = 1 : \sqrt{2}$, or approximately as 5 : 7.

[Sug. FB is a mean proportional between DB and AB. Therefore $\overline{FB}^2 = \frac{1}{3}\overline{AB}^2$. (§ 318.) In like manner, $\overline{FA}^2 = \frac{2}{3}\overline{AB}^2$, etc.]

Ex. 7. Show how the diameter of the earth may be determined by the following method: Drive three stakes into a level piece of ground (or into a shallow piece of water) in line, each two successive stakes being a mile apart, and let each stake project the same distance above the ground (or water). By use of a leveling instrument, determine the amount by which the middle stake projects above a horizontal line connecting the tops of the end stakes. This distance will be found to be 8 in.

[Sug. Use § 318. An arc a mile long on the earth's surface may be taken as equal to its chord. Then from the diagram of § 318 we obtain the following proportion,

the diameter of the earth: 1 mi. = 1 mi.: 8 in.]

Ex. 8. Also show that the distance that the middle stake projects above the horizontal line connecting the top of the two end stakes varies as the square of the distance between the end stakes. Thus, if the two end stakes were placed three times as far apart as in Ex. 7 (that is, 6 mi. apart instead of 2 mi.), the bulge of the earth between them would be 3^2, or 9, times what it was originally.

Hence determine the projection of the middle stake (or bulge of the earth) when the end stakes are 4 mi. apart. When they are 8 mi. 16 mi. 32 mi.

Ex. 9. At the seashore an observer whose eye was 10 ft. above sea level observed a distant steamboat whose hull was hidden for a height of 12 ft. above water level by the bulge of the earth. About how far off was the steamboat?

Ex. 10. A seaman in a lookout 42 ft. above water level could barely see with a glass the topsail of a distant ship, and estimated this topsail to be 45 ft. above sea level. Estimate the distance of the observed vessel from the seaman.

EXERCISES: GROUP 61

Review Questions

Ex. 1. Are two mutually equiangular polygons necessarily similar? Draw two polygons which illustrate your answer.

Ex. 2. Are two polygons whose corresponding sides are proportional necessarily similar? Draw two polygons which illustrate your answer.

Give a list of the properties proved in Book Three concerning

Ex. 3. Chords in a circle. **Ex. 5.** Polygons.

Ex. 4. Secants and tangents. **Ex. 6.** Triangles.

Ex. 7. In making a drawing of an object, what is meant by a scale of $\frac{1}{50}$? By a scale of 1 in. to 100 ft.?

Ex. 8. On the diagram on page 207, how many lines must be measured in order to determine the others? Illustrate your answer numerically (*i.e.*, express in numbers the lengths of the fewest possible lines and then determine the remaining lines).

Ex. 9. What is the efficiency value of Prop. V? Of Prop. XIII?

Ex. 10. On the diagram of Prop. XVI, how many of the line-segments AF, FB, DF, FC, AB, CD, must be measured in order to determine the others? What, then, is the efficiency value of this theorem?

Ex. 11. Make up and answer an example similar to Ex. 10 concerning Prop. XVII.

BOOK FOUR

AREAS OF POLYGONS

328. A **unit of surface** is a square whose side is a **unit** of length; as a square inch, a square yard, or a square centimeter.

The **area of a surface** is the number of units of surface which the given surface contains.

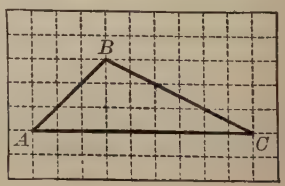
329. By the **area of a polygon** is meant the area of that portion of a plane inclosed by the polygon.

330. Efficiency principle.—It is important for the student to grasp firmly the fact that *area* means not merely a vague largeness of surface, but that it is a *number*. Being a number, it can be resolved into factors; it may be determined as a product of simpler numbers; and it may be handled with ease and precision in various ways. (See Ex. 1, p. 7.)

EXERCISES: GROUP 62

Ex. 1. Draw a rectangle 6 inches long and 4 inches wide, and divide the rectangle into inch squares by drawing lines parallel to the sides. Obtain the area of the rectangle by counting the number of small squares composing the figure. Do you know of any shorter way of getting the area of the rectangle?

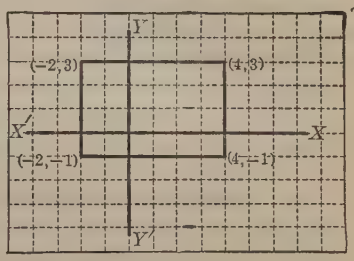
Ex. 2. Obtain the area of the triangle ABC by counting the number of small squares composing it, piecing together the parts of the squares. Do you know of any shorter way of getting the area of the triangle ABC?



On squared paper, locate the triangle or quadrilateral whose vertices are as follows, and determine the area of each figure.

Each pair of numbers inclosed in parenthesis represents a point. Thus in Ex. 3, the first vertex is represented by (4, 3), in which the first number, 4, represents a distance of four spaces to the right of the vertical line YY' (see diagram of Ex. 3), and the number 3, a distance of three spaces above the horizontal line XX'. If the first number in a parenthesis is negative, the number represents a distance to the left of the line YY'; and if the second number is negative, it represents a distance below the line XX'.

Ex. 3. (4, 3), (4, −1), (−2, 3), (−2, −1).



(Ex. 3)

Ex. 4. (1, 1), (1, 5), (3, 3).

Ex. 5. (0, 0), (4, 4), (6, 0), (10, 4).

Ex. 6. (0, 4), (−3, 1), (6, 1).

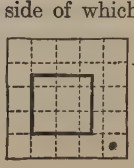
Ex. 7. (4, 5), (8, 1), (−1, 5), (−3, 1).

Ex. 8. (2, 3), (5, 0), (2, −6), (−1, 0).

Ex. 9. (1, 0), (4, 3), (7, −3), (4, −6).

Ex. 10. (2, 6), (10, 2), (4, −4), (−2, 2).

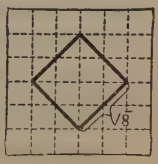
Ex. 11. On squared paper construct a square each side of which is $2\frac{1}{2}$ linear spaces, as on the adjoining diagram. Determine the number of small squares composing it (that is, its area) by counting the small squares. Then determine the area by multiplying $2\frac{1}{2}$ by $2\frac{1}{2}$. Compare the amount of work in the two methods of determining the area of the square.



Ex. 12. On squared paper, construct a square each side of which is 3.5 and determine the area of the square.

Ex. 13. Treat in like manner a square each side of which is $2\frac{1}{4}$. $3\frac{3}{4}$.

Ex. 14. On squared paper, construct a square each side of which is $\sqrt{8}$, as in the adjoining diagram, and determine its area by counting the small squares. Then determine the area of the square by multiplying $\sqrt{8}$ by $\sqrt{8}$. Compare the amount of work in the two methods.



Ex. 15. Treat in like manner a square whose side is $\sqrt{32}$. $\sqrt{18}$.

Ex. 16. Also a rectangle whose sides are $\sqrt{8}$ and $\sqrt{2}$. $\sqrt{18}$ and $\sqrt{8}$.

Ex. 17. Draw two rectangles which are equal in area, but which are not congruent. Also two triangles.

Ex. 18. In Ex. 2, what is the area represented by the triangle ABC, if each linear space on the squared paper represents 1 yard? If it represents 1 mile? 10 miles?

Ex. 19. By first constructing an equilateral triangle, and then bisecting one of its angles, construct a right triangle whose hypotenuse is 2 in. and whose acute angles are 30° and 60°. In the right triangle, how long is the side opposite the angle of 30°?

Ex. 20. $ABCD$ is a quadrilateral inscribed in a circle. $\overset{\frown}{AB}=92°$, $\overset{\frown}{BC}=48°$, $\overset{\frown}{CD}=\frac{1}{4}\overset{\frown}{AD}$. Find all the angles of the figure. Draw AC and BD. How many similar triangles are formed?

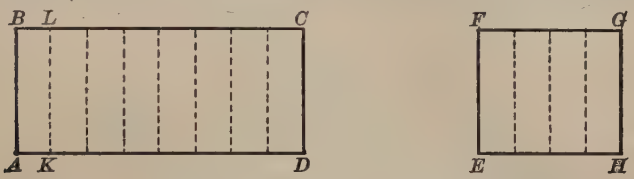
331. Fundamental principle of areas.—The preceding examples illustrate the fact that the *area of a rectangle equals the product of the number of units of length in the base by the number in the altitude.* (Or we may say more briefly that, in a rectangle, the area-number equals the length-number multiplied by the width-number.) Some of the examples also illustrate the fact that this is true when the numbers involved are fractional or irrational, as well as when they are integers. We shall now treat the matter of areas from another point of view, for the sake of special advantages thus obtained.

332. Certain abbreviations are often used when speaking of areas. Thus, instead of "area of a rectangle," for example, it is often convenient to say simply "rectangle." Similarly, instead of "the number of linear units in the base," we may say "the base." In like manner, for "product of the number of linear units in the base by the number of linear units in the altitude," a common abbreviation is "product of the base by the altitude."

Sometimes the word "equivalent" is used to mean "equal in area."

Proposition I. Theorem

333. *The areas of two rectangles having equal altitudes are to each other as their bases.*



Given the rectangles $ABCD$ and $EFGH$, having their altitudes AB and EF equal.

To prove $ABCD : EFGH = AD : EH$.

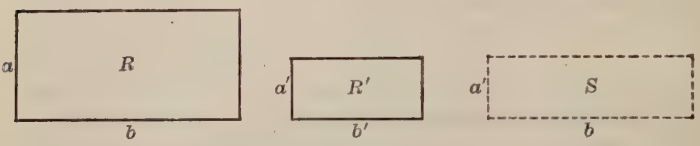
Proof. 1. Take some common unit of measure of AD and EH, as AK, and let it be contained in AD m times and in EH n times; then $\frac{AD}{EH} = \frac{m(AK)}{n(AK)} = \frac{m}{n}$.	1. § 232.
2. Through the points of division of the bases of the two rectangles draw lines $\perp$ the bases.	2. § 89.
3. These lines will divide $ABCD$ into m, and $EFGH$ into n small rectangles, all equal.	3. §§ 97, 149, 164.
4. Hence $\frac{ABCD}{EFGH} = \frac{m(ABLK)}{n(ABLK)} = \frac{m}{n}$.	4. Why?
5. $\therefore \frac{ABCD}{EFGH} = \frac{AD}{EH}$.	5. Why? Q.E.D.

If in the above proof the two lines AD and EH should be found to have no common unit of measure, like AK, the above theorem may still be proved true by a method of proof called the *method of limits*, which is beyond the scope of this book.

334. Cor. *The areas of two rectangles having equal bases are to each other as their altitudes.*

Proposition II. Theorem

335. *The areas of two rectangles are to each other as the products of their bases and their altitudes.*



Given the rectangles R and R', having the bases b and b', and the altitudes a and a', respectively.

To prove $\frac{R}{R'}=\frac{ab}{a'b'}$.

Proof.

1. Construct a rectangle, S, having its base equal to that of R, and its altitude equal to that of R'.	1. § 89, Post. 2.
2. Then $\frac{R}{S}=\frac{a}{a'}$.	2. § 334.
3. Also $\frac{S}{R'}=\frac{b}{b'}$.	3. § 333.
4. Hence $\frac{R}{R'}=\frac{ab}{a'b'}$.	4. Ax. 4.

Q.E.D.

Ex. 1. Find the ratio of the area of a rectangle whose dimensions are 12×8 in. to that of one whose dimensions are 9×2 in.

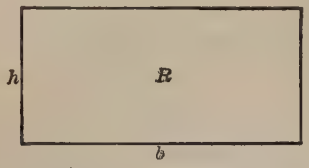
Ex. 2. How many bricks, each 8×5 in., will it take to cover a pavement 60×9 ft.?

Ex. 3. Find the ratio of the areas of two rectangles, one of which is 1.2 ft. by 4.2 ft., and the other 1.68 ft. by 10.24 ft.

Ex. 4. The ratio of the areas of two rectangles is 2 to 5. The base of the first rectangle is 12.8 feet, and the altitude is 4.4 feet. If the altitude of the second rectangle is 8.2 feet, find its base.

Proposition III. Theorem

336. *The area of a rectangle is equal to the product of its base and its altitude.*



Given the rectangle R, with a base containing b, and an altitude containing h units of linear measure.

To prove area of $R = bh$.

Proof. 1. Let U be a square each side of which contains 1 linear unit; then U is the unit of surface.	1. § 328.
2. $\therefore \frac{R}{U} = \frac{bh}{1\times 1} = bh.$	2. § 335.
3. But $\frac{R}{U}$ is the area of R.	3. § 328.
4. Hence area of $R = bh$.	4. Ax. 1. Q.E.D

337. Efficiency value.—By use of this theorem, the problem of finding the area of a rectangle is reduced to the simpler problem of measuring the two linear dimensions of the rectangle and taking their product. Hence the statement in the theorem of § 336 may be taken as the definition of the area of a rectangle, if the teacher so desires.

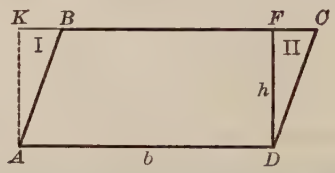
Ex. 1. Find the area of a rectangle whose base is $8\frac{3}{4}$ yd. and whose altitude is $2\frac{1}{2}$ yd.

Ex. 2. The area of a given rectangle is 54.95 sq. rd. and the altitude is 3.14 rd. Find the base of the rectangle.

Ex. 3. The perimeter of a given square is 1 ft. Find the area of the square in square inches.

Proposition IV. Theorem

338. *The area of a parallelogram is equal to the product of its base by its altitude.*



Given the $\square\ ABCD$ with the base AD (denoted by b) and the altitude DF (denoted by h).

To prove area of $ABCD = bh$.

Proof. 1. From A draw $AK \parallel DF$, and meeting CB produced at K.	1. § 100, Post. 2.
2. Then $AK \perp CK$.	2. § 105.
3. $\therefore AKFD$ is a rectangle with base b and altitude h.	3. § 149.
4. In the rt. $\triangle I$ and II, $AB = DC$, and $AK = DF$.	4. Why?
5. Hence $\triangle I \cong \triangle II$.	5. Why?
6. But $AKCD = AKCD$.	6. Ident.
7. $\triangle I \cong \triangle II$.	7. Step 5.
8. $\square ABCD$ = rectangle $AKFD$.	8. Ax. 3.
9. But area of rectangle $AKFD = bh$.	9. Why?
10. $\therefore$ area of $\square ABCD = bh$.	10. Why? **Q.E.D.**

339. Cor. 1. *Two parallelograms which have equal bases and equal altitudes are equal in area.*

340. Cor. 2. *Two parallelograms which have equal bases are to each other as their altitudes;*

Two parallelograms which have equal altitudes are to each other as their bases.

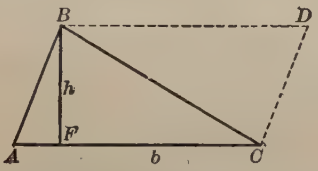
341. Cor. 3. *Any two parallelograms are to each other as the products of their bases and altitudes.*

Ex. 1. The base of a parallelogram is 1 ft. 8 in. and the altitude is 1 ft. Find the area of the parallelogram in square inches.

Ex. 2. The area of a parallelogram is 255.78 sq. ft. and the base is 12.6 ft. Find the altitude.

Proposition V. Theorem

342. *The area of a triangle is equal to half the product of its base by its altitude.*



Given the $\triangle ABC$ with the base AC (denoted by b), and the altitude FB (denoted by h).

To prove area of $\triangle ABC = \frac{1}{2}bh$.

Proof. 1. Draw $BD \parallel AC$, and $CD \parallel AB$.	1. § 100.
2. Then $ABDC$ is a parallelogram and BC is one of its diagonals.	2. § 147.
3 $\therefore \square ABDC = 2\triangle ABC$.	3. § 154.
4. But area of $\square ABDC = bh$.	4. Why?
5. Hence $2\triangle ABC = bh$.	5. Ax. 1.
6. $\therefore$ area of $\triangle ABC = \frac{1}{2}bh$.	6. Ax. 5.

Q.E.D.

343. Cor. 1. *A parallelogram is equivalent to twice a triangle having the same base and altitude.*

344. Cor. 2. *Two triangles which have equal bases and equal altitudes (or equal bases in the same straight line and their vertices in a line parallel to the base) are equal in area.*

345. Cor. 3. *Two triangles which have equal bases are to each other as their altitudes;*

Two triangles which have equal altitudes are to each other as their bases.

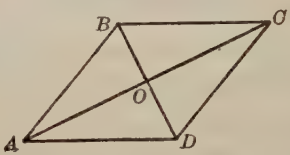
346. Cor. 4. *Any two triangles are to each other as the products of their bases and altitudes.*

Ex. 1. Find the area of a triangle whose base is 17.06 in. and whose altitude is 8.9 in.

Ex. 2. The area of a given triangle is 180 sq. in. and the base is 1 ft. 3 in. Find the altitude in feet and inches.

Proposition VI. Theorem

347. *The area of a rhombus is equal to half the product of the diagonals of the rhombus.*



Given the rhombus $ABCD$ with the diagonals AC and BD.

To prove area of $ABCD = \frac{1}{2}AC \times BD$.

Proof. 1.	$AB = BC$.	1. § 148.
2	$AD = DC$.	2. Why?
3.	$\therefore BD \perp AC$.	3. § 128.
4.	Area of $\triangle ABC = \frac{1}{2}AC \times BO$.	4. § 342.
5.	Area of $\triangle ADC = \frac{1}{2}AC \times OD$.	5. Why?
6.	$\therefore \triangle ABC + \triangle ADC = \frac{1}{2}AC(BO + OD)$.	6. Why?
7.	$\therefore$ Area of $ABCD = \frac{1}{2}AC \times BD$.	7. Ax. 7.

Q.E.D.

348. Cor. *If the diagonals of a quadrilateral are perpendicular to each other, the area of the quadrilateral is equal to half the product of the diagonals.*

Ex. 1. The diagonals of a rhombus are 1 yd. and 1 ft. Find the area of the rhombus in square feet.

Ex. 2. The area of a given rhombus is 256 sq. in., and one diagonal is double the other. Find the diagonals.

EXERCISES: GROUP 63

Ex. 1. D is the midpoint of the base AC of the triangle ABC. The altitude of the triangle is 10 in. and the base is 14 in. Construct the figure and find the area of the triangle ABD, and of the triangle DBC.

Ex. 2. In the triangle PQR, T is the midpoint of PR. Prove the triangles PQT and TQR equal in area.

Ex. 3. If the area of a triangle is 96 sq. in. and its altitude is 1 ft., find the base in feet and inches.

Ex. 4. The base of a given triangle is 14.4 ft. and the altitude is 3.2 ft. Find the side of a square whose area is equal to that of the given triangle.

Ex. 5. One side of a rhombus is 14 in. and one of its angles is 30°. Find the area of the rhombus. (See Ex. 7, p. 104.)

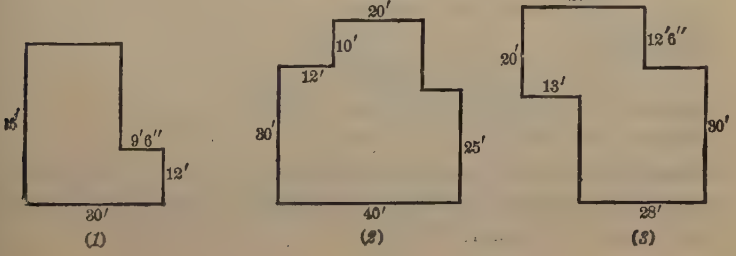
Ex. 6. Two sides of a parallelogram are 8 in. and 12 in. and the angle included by them is 30°. Find the area of the parallelogram.

Ex. 7. Two sides of a triangle are 10 in. and 12 in. and the angle included by them is 30°. Find the area.

Ex. 8. Find the area of a square inscribed in a circle whose radius is 10 in.

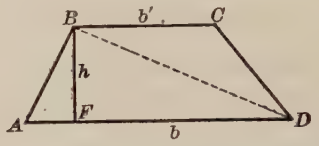
Ex. 9. FC and AD are parallel lines. H and B are points on FC such that $AFHD$ and $ABCD$ are parallelograms. Prove the triangles FHD and BCA equal in area.

Ex. 10. Find the area of each of the following cellar floors by dividing them into rectangles.



Proposition VII. Theorem

349. *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*



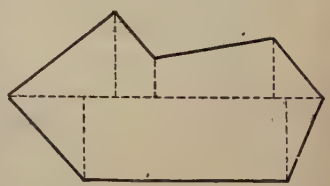
Given the trapezoid $ABCD$ with the bases AD and BC (denoted by b and b'), and the altitude FB (denoted by h).

To prove area of $ABCD = \frac{1}{2}(b+b')h$.

Proof. 1. Draw the diagonal BD.	1. Post. 1.
2. $ABCD = \triangle ABD + \triangle BCD$.	2. Ax. 7.
3. h is the altitude of both $\triangle ABD$ and $\triangle BCD$.	3. § 156.
4. Area of $\triangle ABD = \frac{1}{2}bh$.	4. Why?
5. Area of $\triangle BCD = \frac{1}{2}b'h$.	5. Why?
6. $\therefore \triangle ABD + \triangle BCD = \frac{1}{2}(b+b')h$.	6. Ax. 2.
7. $\therefore$ area of $ABCD = \frac{1}{2}(b+b')h$.	7. Ax. 1.

Q.E.D.

350. The area of a polygon of four or more sides can usually be found in one of several ways; as:



By dividing the polygon into triangles and taking the sum of the areas of the triangles; or,

By drawing the longest diagonal of the polygon and drawing perpendiculars to this diagonal from the vertices which it does not meet, and obtaining the sum of the areas of the triangles, rectangles, and trapezoids thus formed.

EXERCISES: GROUP 64

Ex. 1. The bases of a trapezoid are 3 ft. 6 in. and 2 ft. 8 in., and the altitude is 2 ft. 2 in. Find the area in square inches.

Ex. 2. If the area of a trapezoid is 135, and its bases are 12 and 18, find its altitude.

Ex. 3. Find the altitude of a trapezoid in which the bases are 1.4 ft. and 2.5 ft., and the area is 4.368 sq. ft.

Ex. 4. By solving the formula $A = \frac{1}{2}(b+b')h$, find the value of b in terms of the other letters.

Ex. 5. The cross section of a railroad cutting is a trapezoid; the width at the bottom is 12.4′, the width at the top, 32.4′, and the depth, 7.5′. Find the area of the cross section.

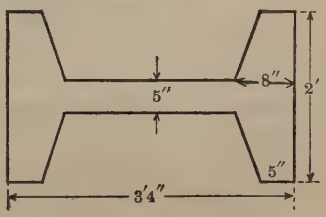
Ex. 6. The bases of a trapezoid are 14 and 32. One leg is 20 and the angle at the base adjacent to this leg is 30°. Find the area of the trapezoid.

Ex. 7. Prove that if a square and a rhombus with oblique angles stand on the same base, the area of the square is greater.

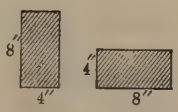
Ex. 8. ABC is a triangle and D is the midpoint of the side BC. G is any point on AD. Prove that the triangles ACG and AGB are equal in area.

Ex. 9. Can two squares with different perimeters have the same area? Give a numerical illustration of your answer.

Ex. 10. Find the area of the adjoining figure.



Ex. 11. The supporting power of a wooden beam (of rectangular cross section and of given length) varies as the area of the cross section multiplied by the height of the beam. If the cross section of a given beam is 4″×8″, compare the supporting power of the beam when it rests on the narrow edge (4″) with its supporting power when it rests on its wide edge (8″).



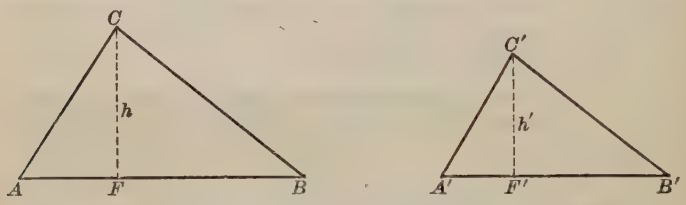
Ex. 12. The diagonal PR of the parallelogram $PQRS$ is produced to the point T. Prove that the triangles PQT and PST are equal in area.

Ex. 13. On squared paper, locate the polygon whose vertices are (0, 0), (3, 6), (7, 2), (10, 8), (12, 0), (8, −4), (4, −4), and determine its area.

What is this area, if one linear space on the squared paper represents 10 ft.?

Proposition VIII. Theorem

351. *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



Given the similar $\triangle ABC$ and $A'B'C'$ with AB and $A'B'$ corresponding sides.

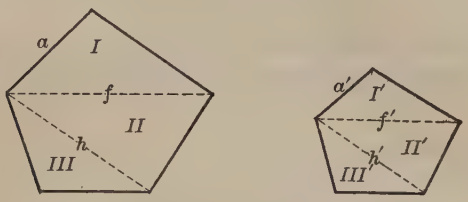
To prove $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Proof. 1. Draw the corresponding altitudes h and h'.	1. § 131.
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times h}{A'B' \times h'} = \dfrac{AB}{A'B'} \times \dfrac{h}{h'}$.	2. § 346.
3. But $\dfrac{h}{h'} = \dfrac{AB}{A'B'}$.	3. § 314.
4. Substituting $\dfrac{AB}{A'B'}$ for its equal $\dfrac{h}{h'}$, $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	4. Ax. 9.
	Q.E.D.

Ex. If a pair of corresponding sides of two similar triangles are 4 ft. and 5 ft., find the ratio of the areas of the triangles.

Proposition IX. Theorem

352. *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*



Given two similar polygons with their areas denoted by A and A', and a and a' any pair of corresponding sides.

To prove $A : A' = a^2 : a'^2$.

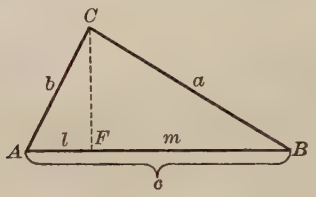
Proof. 1. From any pair of corresponding vertices, draw the corresponding diagonals f and f', and h and h'.	1. Post. 1.
2. These diagonals will divide the polygons into pairs of similar triangles, denoted by I, I'; II, II'; and III, III', respectively.	2. § 310.
3. But $\frac{I}{I'}=\frac{a^2}{a'^2}$.	3. § 351.
4. Then $\frac{I}{I'}=\left(\frac{f^2}{f'^2}\right)=\frac{II}{II'}=\left(\frac{h^2}{h'^2}\right)=\frac{III}{III'}$.	4. § 351, Ax. 1.
5. $\therefore \frac{I}{I'}=\frac{II}{II'}=\frac{III}{III'}$.	5. Ax. 1.
6. Hence $\frac{I+II+III}{I'+II'+III'}=\frac{I}{I'}=\frac{a^2}{a'^2}$.	6. § 286, Ax. 1.
7. $\therefore \frac{A}{A'}=\frac{a^2}{a'^2}$.	7. Ax. 7.

Q.E.D.

353. Cor. *The areas of two similar polygons are to each other as the squares of their perimeters, or of any two of their corresponding lines.*

Proposition X. Theorem

354. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*



Given the rt. $\triangle ABC$ with the hypotenuse AB denoted by c, and the legs by a and b.

To prove $c^2=a^2+b^2$.

Proof. 1. Draw $CF \perp AB$, and denote AF by l and FB by m.		1. § 131.
2.	$cm=a^2$.	2. § 319.
3.	$cl=b^2$	3. § 319.
4.	$cm+cl=a^2+b^2$,	4. Ax. 2.
or	$c(m+l)=a^2+b^2$.	
5. But	$m+l=c$.	5. Ax. 7.
6.	$\therefore c^2=a^2+b^2$.	6. Ax. 9.

Q.E.D.

355. Cor. *The square of either leg of a right triangle is equal to the square of the hypotenuse diminished by the square of the other leg.*

Ex. 1. Find the hypotenuse of a right triangle in which the legs are 12 in. and 5 in.

Ex. 2. A certain ladder is 17 ft. long and is placed with its foot 8 ft. from the bottom of a building. How far up the side of the building will the top of the ladder reach?

Ex. 3. Find the area of a square whose diagonal is 10 in.

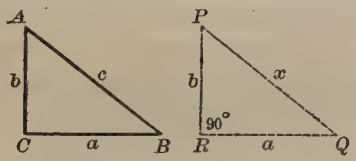
Ex. 4. Two sides of a parallelogram are 20 ft. and 10 ft., and the angle between them is 60°. Find the area of the parallelogram.

Ex. 5. Two sides of a triangle are 24 in. and 18 in., and the angle between them is 45°. Find the area of the triangle.

Ex. 6. Find the area of a rectangle whose diagonal is b, and one of whose sides is a.

Ex. 7. Two poles are 20 ft. and 38 ft. high, respectively. The distance between the poles is 60 ft. Find the distance between their tops.

Ex. 8. If the sum of the squares of two sides of a triangle equals the square of the other side, the triangle is a right triangle.



[Suggestion. **Given** $\triangle ABC$ with $a^2+b^2=c^2$.

To prove $\angle C=$ a rt. $\angle$.

Proof.

1. Construct a $\triangle PQR$ in which $\angle R$ is a right $\angle$, $RP=b$, and $RQ=a$.	1. How?
2. In $\triangle RPQ$, $a^2+b^2=x^2$.	2. § 354.
3. But $a^2+b^2=c^2$.	3. Hyp., etc.]

Ex. 9. The three sides of each of three triangles are given as follows: 7, 8, 15; 5, 14, 13; 1, 2, $\sqrt{3}$. Is there a right triangle among these triangles?

Ex. 10. A pair of corresponding sides of two similar polygons are 13 ft. and 24 ft. The area of the first polygon is 1352 sq. ft. Find the area of the second.

Ex. 11. The areas of two similar polygons are 100 and 256. One side of the first is 8. Find the corresponding side of the second polygon.

Ex. 12. In a circle whose radius is 10 in., find (in radical form) the length of a chord of 45°.

Ex. 13. In a circle whose radius is 10, OA and OB are radii perpendicular to each other. Find (in radical form) the length of OF, the perpendicular from O to the chord AB.

Ex. 14. In a given circle, OA and OB are radii perpendicular to each other. The chord AB is 8 in. long. Find OA.

Ex. 15. One leg of a right triangle is 6 in. and the adjacent acute angle is 30°. Find the other two sides and the area.

Ex. 16. The hypotenuse of a right triangle is 10 in. and one acute angle is 45°. Find the other sides and the area.

Ex. 17. One leg of a right triangle is 4 and the angle opposite is 30°. Find the other sides and the area.

Ex. 18. The base angles of a trapezoid are 30° and 45°. The leg adjacent to the first of these angles is 12 ft. and the upper base is 10 ft. Find the altitude and the area.

Ex. 19. In an isosceles trapezoid the angles at the base are each 45°, and each leg is 10 ft. The upper base is 12 ft. Find the area.

Ex. 20. In a given trapezoid the upper base is 10, the altitude is 8, and the base angles are 45° and 60°. Find the area.

Ex. 21. From a point P exterior to a circle tangents are drawn. The distance of P from the center of the circle is 4 in., and the length of each of the tangents is 2 in. Find the diameter of the circle, and the angle between the two tangents.

Ex. 22. In a circle a triangle ABC is inscribed. A straight line is drawn through A cutting BC at E and the circle at D. Prove the triangles CDE and ABE are similar. If $DE=2$ in., and $BE=3$ in., find the ratio of the areas of the two triangles.

Ex. 23. In a quadrilateral $ABCD$, $AB=AD$ and $CB=CD$. If $BD=x$ and $AC=y$, obtain the area of $ABCD$ in terms of x and y. Give reasons for your answer.

Ex. 24. If the perimeter of a rhombus is 20 in. and one of the diagonals is 6 in., find the area of the rhombus.

Ex. 25. In the triangle ABC, $AB=6$, $BC=7$, and $CA=8$. Through a point D on AB and two units from B, a line is drawn parallel to BC. Find the ratio of the areas of the two parts into which the triangle is divided.

EXERCISES: GROUP 65

Theorems Concerning Areas

Ex. 1. Any straight line drawn through the point of intersection of the diagonals of a parallelogram divides the parallelogram into two parts which are equal in area.

Ex. 2. If, in the triangle ABC, D and F are the midpoints of the sides AB and AC, respectively, the area of triangle ADF equals one fourth the area of triangle ABC. (Use § 351.)

Ex. 3. If the midpoints of two adjacent sides of a parallelogram are joined, the area of the triangle so formed equals one eighth the area of the parallelogram.

Ex. 4. If, in the triangle ABC, D and F are the midpoints of the sides AB and AC, respectively, the triangles ADC and AFB are equal in area.

Ex. 5. In a right triangle show, by obtaining expressions for the area of the figure, that the product of the legs equals the product of the hypotenuse by the altitude upon the hypotenuse.

Ex. 6. If two triangles are equal in area, and the altitude of one is three times the altitude of the other, find the ratio of their bases.

Ex. 7. If two isosceles triangles have their legs equal, and if half the base of one is equal to the altitude of the other, the triangles are equal in area.

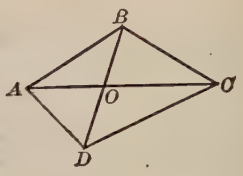
Ex. 8. The area of an isosceles right triangle equals one fourth the square of the hypotenuse of the right triangle.

[Suɢ. Denote the hypotenuse by h and a leg by x. Then show that $\frac{1}{2}x^2 = \frac{1}{4}h^2$.]

Ex. 9. The line joining the midpoints of the parallel sides of a trapezoid divides the trapezoid into two parts equal in area.

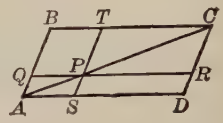
Ex. 10. $PQRS$ is a trapezoid in which the side QR is parallel to PS. The diagonals QS and PR intersect at O and are perpendicular to each other. QPS is a right angle. If PQ is denoted by a, PS by b, PR by c, and OS by x, prove $x = ab/c$.

Ex. 11. Given AOC and BOD straight lines and $OB=OD$. **Prove the** $\triangle ABC$ and ADC equal in area.



Ex. 12. The lines joining the midpoint of one diagonal of a quadrilateral to the vertices not joined by the diagonal divide the quadrilateral into two parts equal in area.

Ex. 13. Given QR and TS passing through P, any point on the diagonal AC of a $\square$, $QR \parallel AD$, and $TS \parallel AB$. Prove the $\square$ $QBTP$ and $PRDS$ equal in area.



EXERCISES: GROUP 66

Numerical Properties of Lines (or Areas)

Ex. 1. If AD is the altitude of the triangle ABC, $\overline{AB}^2-\overline{AC}^2=\overline{BD}^2-\overline{DC}^2$.

Ex. 2. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides equals the sum of the squares of the other pair of sides.

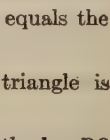
Ex. 3. The square of the altitude of an equilateral triangle is three fourths the square of one side.

Ex. 4. If AB is the hypotenuse of a right triangle, and the leg BC is bisected at K, $\overline{AB}^2-\overline{AK}^2=3\overline{CK}^2$.

Ex. 5. AB is the hypotenuse of a right triangle ABC, P a point on AC and Q a point on BC. Prove $\overline{AQ}^2+\overline{BP}^2=\overline{AB}^2+\overline{PQ}^2$.

Ex. 6. In the right triangle ABC, BE and CF bisect the legs AC and AB in the points E and F. Prove $4\overline{BE}^2+4\overline{CF}^2=5\overline{BC}^2$.

Ex. 7. Prove geometrically that $(a+b)^2=a^2+b^2+2ab$.



Ex. 8. Similarly, prove $(a-b)^2=a^2+b^2-2ab$.

Ex. 9. Similarly, prove $(a+b)(a-b)=a^2-b^2$.

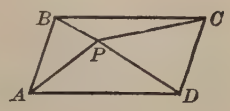
Ex. 10. Two beams of the same length and material have cross sections which are $2''\times4''$ and $3''\times8''$, respectively. Find the ratio of the greatest supporting power of the two beams. (See Ex. 11, p. 235.)

Ex. 11. On a given map, a certain country occupies an area of $12\frac{3}{4}$ square inches. If the scale of the map is 1 inch to 50 miles, find the area of the country in square miles.

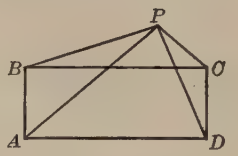
EXERCISES: GROUP 67

Use of Auxiliary Lines

Ex. 1. Given $ABCD$ a $\square$ and P any point inside $ABCD$. Prove $\triangle PAD + \triangle PBC = \triangle PAB + \triangle PCD$.

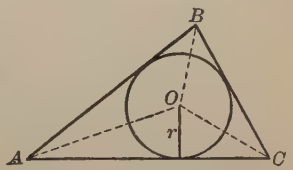


Ex. 2. Given $ABCD$ a rectangle. Prove $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.



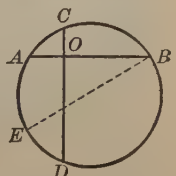
Ex. 3. The area of a triangle is equal to one half the product of its perimeter by the radius of the inscribed circle.

[Sug. If the $\triangle$ is ABC and O is the center of the inscribed circle, draw OA, OB, OC, and find the sum of the areas of the $\triangle AOC$, AOB, BOC.]



Ex. 4. If the extremities of one leg of a trapezoid are joined to the midpoint of the other leg, the middle one of the three triangles thus formed is equal in area to half the trapezoid.

Ex. 5. The area of a trapezoid is equal to the product of one leg by the perpendicular on that leg from the midpoint of the other leg.



Ex. 6. Given the chords AB and CD perpendicular to each other and intersecting at O. Prove $\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2 + \overline{OD}^2 = (\text{diameter})^2$.

[Sug. Draw the diameter BE and the chords AC, BD, DE. Prove $AC = ED$, etc.]

Ex. 7. If the midpoints of the sides of a quadrilateral are joined in order, the parallelogram thus formed is equal in area to one half the quadrilateral.

Ex. 8. A quadrilateral is equal in area to a triangle two of whose sides are equal to the diagonals of the quadrilateral, the angle included by these sides being equal to one of the angles formed by the intersection of the diagonals.

EXERCISES: GROUP 68

THEOREMS PROVED BY VARIOUS METHODS

Ex. 1. If through the midpoint of one leg of a trapezoid a line is drawn parallel to the other leg to meet one base and the other base produced, the parallelogram so formed is equal in area to the trapezoid.

Ex. 2. If the midpoints of two sides of a triangle are joined to any point in the base, the quadrilateral so formed is equal in area to half the triangle.

Ex. 3. If P is any point on AC, the diagonal of a parallelogram $ABCD$, the triangles APB and APD are equal in area.

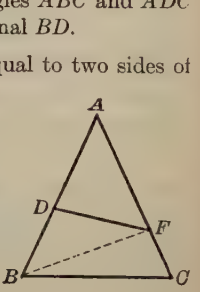
Ex. 4. If the side of an equilateral triangle is denoted by a, the area of the triangle equals $\frac{a^2\sqrt{3}}{4}$.

Ex. 5. Find the ratio of the areas of two equilateral triangles, if the altitude of one equals the side of the other.

Ex. 6. If perpendiculars are drawn from any point within an equilateral triangle to the three sides, their sum is equal to the altitude of the triangle.

Ex. 7. If, in the quadrilateral $ABCD$, the triangles ABC and ADC are equal in area, the diagonal AC bisects the diagonal BD.

Ex. 8. If two triangles have two sides of one equal to two sides of the other, and the included angles supplementary, the triangles are equal in area.



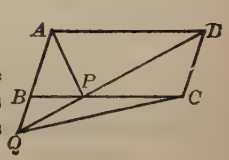
Ex. 9. If two triangles have an angle of one equal to an angle of the other, the areas of the triangles are to each other as the products of the sides including the equal angles.

[SUG. Let ABC and ADF be the given triangles.

Then it is required to prove that $\frac{\triangle ADF}{\triangle ABC} = \frac{AD \times AF}{AB \times AC}$.

Draw BF, and compare each of the given $\triangle$ with ABF. See § 345.]

Ex. 10. P is any point in the side BC of the parallelogram $ABCD$, and DP produced meets AB produced in Q. Show that the triangles BPA and CPQ are equal in area.



EXERCISES: GROUP 69

Problems in Constructing Areas

Ex. 1. Construct a square equal to the sum of two given squares.

Ex. 2. Construct a square equal to the difference of two given squares.

[Sug. A side of the larger of the two given squares may be taken as the hypotenuse of a right triangle of which a side of the other given square is one leg.]

Ex. 3. Construct a square having twice the area of a given square.

Ex. 4. Construct a square having three times the area of a given square.

Ex. 5. Construct a square equal to the sum of three given squares.

Ex. 6. Transform a given triangle into an isosceles triangle having the same base and area.

Ex. 7. Transform a given triangle into a triangle having the same base and area, but having a given angle adjacent to the base.

Ex. 8. Transform a given triangle into a triangle with the same base and area, but having another given side.

Ex. 9. Transform a given parallelogram into a parallelogram having the same base and area, but containing a given angle.

Ex. 10. Construct a triangle similar to a given triangle and containing twice the area.

How is the construction changed to construct a similar triangle containing five times the area?

Ex. 11. Bisect the area of a given triangle by a line parallel to the base.

Ex. 12. Bisect a parallelogram by a line perpendicular to the base.

Ex. 13. Through any given point draw a line bisecting the area of a given parallelogram.

Ex. 14. Transform a given rectangle into a rectangle having the same area and a given base.

EXERCISES: GROUP 70

Practical Applications

Ex. 1. Find the ratio between the supporting power of a $2'' \times 6''$ beam when placed on a $6''$ side and the supporting power of the same beam when placed on a $2''$ side. Treat similarly a beam $a'' \times b''$. (See Ex. 11, p. 235.)

Ex. 2. How many beams, each $1\frac{1}{2}'' \times 6''$, must be placed on their wide edges to equal in supporting power one beam of the same size when placed on its narrow edge?

Ex. 3. Given a log 12 in. in diameter, find the ratio of the strength of a beam cut from it by the method described in Ex. 6, p. 222, to the strength of a square beam cut from the same log.

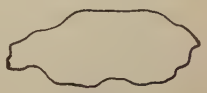
Ex. 4. Find the ratio of the strength of a beam cut from the same log by the method of Ex. 6, p. 222, to the strength of a beam cut so that its width equals $\frac{1}{4}$ the diameter of the log.

Ex. 5. When an irregular area like $ABCD$ is calculated by means of equidistant offsets (like AB, DC, and the lines $\parallel$ to them in diagram), the following rule is used:

To half the sum of the initial and final offsets add the sum of all the intermediate offsets, and multiply the sum by the common distance between the offsets.

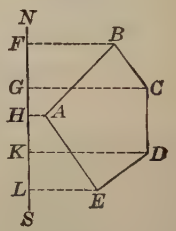
Prove this rule.

(Ex. 5)


(Ex. 6)

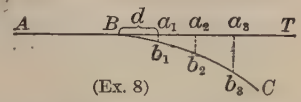
Ex. 6. Show how the rule of Ex. 5 could be used to calculate an area whose entire boundary is an irregular curved line.

Ex. 7. Surveyors often determine the area of a piece of land, as of $ABCD$, by taking an auxiliary line as NS, measuring the perpendicular distances from A, B, C, D, E to NS, and the intercepts on NS between these perpendiculars, and then combining the areas of the various trapezoids, rectangles, or triangles, formed. Supply probable numbers for the lengths of lines on the diagram, and compute the area of $ABCDE$.



(Ex 7)

Ex. 8. Frequently (as in the case where the center of the curve cannot be seen from the curve) a railroad curve is laid out by constructing a series of equidistant offsets perpendicular to the tangent of the curve.



(Ex. 8)

Thus, if ABT is a straight track and BC is a curve to be laid out tangent to AB at B, mark off Ba_1, a_1a_2, a_2a_3, all $=d$, and construct the $\perp$ offsets a_1b_1, a_2b_2, a_3b_3 by using the formula $a_nb_n = r - \sqrt{r^2 - n^2d^2}$, where r is the radius of the curve. Prove that this formula is correct.

Ex. 9. A steamboat is traveling at the rate of 12 mi. an hour, and a boy walks across her deck at right angles to her line of motion at the rate of 3 mi. an hour. Draw a diagram to show the direction of the boy's resultant motion and determine his speed.

Ex. 10. Make up and work a similar example concerning a mail bag thrown from a train.

Ex. 11. Also concerning a breeze blowing into a window of a moving trolley car.

Ex. 12. Two forces, one of 300 lb., the other of 400 lb., act at right angles on the same body. Find their resultant.

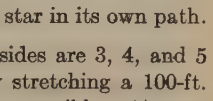
Ex. 13. Make up and work an example similar to Ex. 12.

Ex. 14. A river is flowing at a rate of 4.25 mi. per hour, and a man is rowing at right angles to the current at a rate of 3.75 mi. per hour. What is the resultant velocity of the man?

Ex. 15. If a star has a velocity of 15 mi. a second toward the earth and a velocity of 20 mi. a second at right angles with a line drawn from the star to the earth, find the velocity of the star in its own path.

Ex. 16. Using the fact that a triangle whose sides are 3, 4, and 5 units of length is a right triangle, show how, by stretching a 100-ft. tape, to construct a right angle as accurately as possible. (Among the ancient Egyptians a class of workmen existed called rope stretchers, whose business it was to construct right angles in this general way.)

Ex. 17. It is customary for carpenters to express the pitch of a roof as the quotient obtained by dividing the height (BD) of the peak above the span (AC) by the span.



Thus, if $AC = 2BD$, the pitch is $\frac{1}{2}$; if $AC = 3BD$, the pitch is $\frac{1}{3}$, etc.

If the pitch of a given roof is $\frac{1}{3}$ and the span is 20 ft., compute the

length of the rafters, projections at the eaves being neglected. Also show how, if the rafters are 8 in. by 2 in., by use of the carpenter's square, the bevel lines at B may be cut. Also those at A.

EXERCISES: GROUP 71

Review Questions

Ex. 1. Construct two triangles which are equal in area but not congruent.

Ex. 2. Construct two rectangles which are equal in area but not congruent.

Give a list of the properties proved in Book Four concerning

Ex. 3. Triangles.

Ex. 4. Rectangles.

Ex. 5. Parallelograms.

Ex. 6. The rhombus.

Ex. 7. The trapezoid.

Ex. 8. Similar polygons.

Ex. 9. In geometry, the expression "the product of the base by the altitude" is an abbreviation for what? In general, "the product of two lines" is an abbreviation for what?

Ex. 10. "The area of a rectangle divided by the base" is an abbreviation for what?

Ex. 11. How many sides of a right triangle is it necessary to measure in order to determine all of the sides of the right triangle? What, then, is the efficiency value of Prop. X?

Ex. 12. State the efficiency value of Prop. IV. Of Prop. V.

Ex. 13. State the efficiency value of Prop. VII. Of Prop. VIII.

Ex. 14. Prop. X is sometimes called the Pythagorean Theorem. Find out and state why it has this name.

Ex. 15. In order to find the area of a quadrilateral whose sides taken in order are a, b, c, and d, the ancient Egyptians used the formula $\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$. State for what classes of quadrilaterals this formula will give the correct area. When the formula is incorrect, does it give an area which is too large or too small?

NUMERICAL TRIGONOMETRY

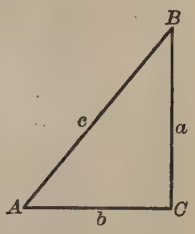
356. Trigonometric ratios.—In solving triangles by numerical trigonometry the ratios termed *sine*, *cosine*, and *tangent* are used. The abbreviations for these words are *sin*, *cos*, and *tan*, respectively.

If ABC is a triangle with a right angle at C,

$$\sin A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}.$$

$$\cos A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

$$\tan A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

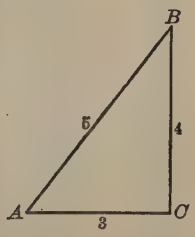


Thus in the adjoining triangle whose sides are 3, 4, and 5,

$$\sin A = \tfrac{4}{5} = .8.$$

$$\cos A = \tfrac{3}{5} = .6.$$

$$\tan A = \tfrac{4}{3} = 1.3333+.$$



Let the student write the values in this triangle of sin B, cos B, tan B.

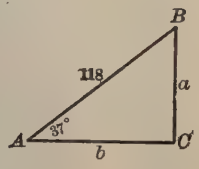
357. Trigonometric tables.—Letting A (one of the acute angles of a right triangle) have successive integral values from 1° to 89°, the numerical values of the trigonometric ratios of each of these angles have been computed (by methods beyond the scope of this book) and arranged in tables. See p. 250.

TRIGONOMETRIC TABLES

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

358. Solution of Triangles.

Ex. 1. In a right triangle, given $A=37°$, $c=118$, find a.



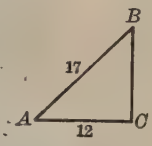
By § 356, $\frac{a}{118}=\sin 37°$.

$\therefore a=118\times\sin 37°$

$=118\times .6018$

$=71.01+$. *Ans.*

Ex. 2. In a right triangle, $b=12$ and $c=17$. Find the angle A to the nearest degree.

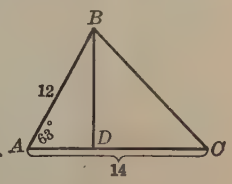


By § 356, $\cos A=\frac{12}{17}=.7059$.

$\therefore A=45°+$. *Ans.*

Ex. 3. Find the area of a triangle in which two of the sides are 12 and 14, and the angle included by these sides is 63°.

Let ABC be the triangle as indicated. We draw the altitude BD and by solving the right triangle ABD, find the altitude h. Thus



$\frac{h}{12}=\sin 63°$.

$\therefore h=12\times\sin 63°=12\times .8910$

$=10.692$.

Then area $\triangle ABC=\frac{1}{2}(14\times 10.692)=74.84+$. *Ans.*

EXERCISES: GROUP 72

Ex. 1. In the right triangle ABC, given $c=212$ and $A=27°$, find a and b.

Ex. 2. In the right triangle ABC, given $a=12$ and $b=17$, find A to the nearest degree.

Ex. 3. When the angle of elevation of the sun is 57°, the shadow of a certain tree is 116 ft. long. Find the height of the tree.

Ex. 4. Find the altitude of an isosceles triangle in which one of the legs is 6.8 and a base angle is 72°.

Ex. 5. In an isosceles triangle the base is 10 and a leg is 12. Find the vertex angle.

Ex. 6. A regular pentagon is inscribed in a circle whose radius is 10. Find a side of the pentagon and also the apothem.

Ex. 7. Find the area of a triangle in which two of the sides are 8.6 and 7.4, and the angle included by these sides is 64.8°.

Ex. 8. Find the area of a parallelogram in which two adjacent sides are 20 and 12 and the angle between them is 38°.

Ex. 9. Find the area of a regular decagon one of whose sides is 8 in.

Ex. 10. Find the area of a trapezoid in which the upper base is 80, the lower base is 120, one leg is 48, and the angle between that leg and the lower base is 64°.

Ex. 11. The diagonals of a parallelogram are 12 and 18 and they intersect at an angle of 37°. Find the area of the parallelogram.

Ex. 12. In the triangle ABC, $AB = 212$, $BC = 324$, and angle $B = 39°$. Find the side AC.

[Sug. Draw the altitude AD, and solve the triangle BAD, and then the triangle ADC.]

GEOMETRY IN DESIGN

All branches of art have used geometry. This section of mosaic floor from the Vatican in Rome shows how colored tiles may be laid in graceful geometric patterns. (*Ewing Galloway*)

DECORATED POTTERY

Vases and plates rescued from ancient ruined or abandoned buildings and cities show that even long ago geometric designs were used to beautify pottery. (*Metropolitan Museum*)

NAVAJO BLANKET

The Indian perhaps knew little of the theorems of geometry, yet he constantly used its figures to adorn his tepee and decorate his blanket. (*American Museum of Natural History*)

BOOK FIVE

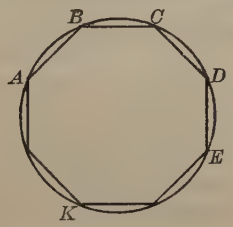
REGULAR POLYGONS; MEASUREMENT OF THE CIRCLE

359. A **regular polygon** is a polygon that is both equilateral and equiangular.

360. Efficiency principle.—Many of the properties of regular polygons are best obtained by the aid of circles. On the other hand, certain important properties of circles are derived to the best advantage by the use of regular polygons.

Proposition I. Theorem

361. *An equilateral polygon inscribed in a circle is a regular polygon.*



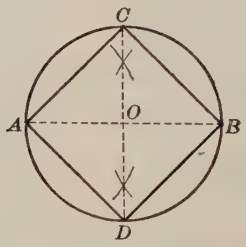
Given $ABC \ldots K$ an inscribed polygon, with its sides AB, BC, CD, etc., equal.

To prove that $ABC \ldots K$ is a regular polygon.

Proof. 1.	$\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$, etc.	1. § 199.
2.	$\therefore \overset{\frown}{ABC} = \overset{\frown}{BCD} = \overset{\frown}{CDE}$, etc.	2. Ax. 2.
3.	$\therefore \angle ABC = \angle BCD = \angle CDE$, etc.	3. § 243.
4.	Hence polygon $ABC \ldots K$ is regular.	4. § 359. **Q.E.D.**

Proposition II. Problem

362. *To inscribe a square in a given circle.*



Given the circle O.

To inscribe a square in the circle O.

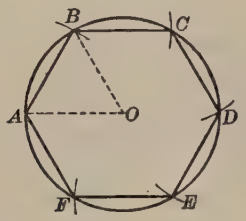
Construction. 1. Draw any diameter, AB, of the circle O.	1. Post. 1.
2. At O construct the line $CD \perp AB$ and meeting the circle at the points C and D.	2. § 89.
3. Draw the chords AC, CB, BD, DA. Then $ACBD$ is the square required.	3. Post. 1.
Proof.	
1. $\angle AOC = \angle COB = \angle BOD = \angle DOA$.	1. § 63.
2. $\overset{\frown}{AC} = \overset{\frown}{CB} = \overset{\frown}{BD} = \overset{\frown}{DA}$.	2. § 196.
3. $\therefore$ chord AC = chord CB, etc.	3. § 201.
4. Hence $ACBD$ is regular; that is, it is a square inscribed in the circle O.	4. §§ 361, 80. Q.E.F.

363. Cor. *By bisecting the arcs AC, CB, BD, etc., of the given circle, and drawing chords, a regular octagon may be inscribed in the circle; by repeating the process, regular polygons of* 16, 32, 64, . . . *and* 2^n *sides may be inscribed, where n is a positive integer greater than* 1.

Ex. In the shortest way, find the ratio of the areas of two squares whose diagonals are 9 in. and 24 in., respectively. (Use § 353.)

Proposition III. Problem

364. *To inscribe a regular hexagon in a given circle.*



Given the circle O.

To inscribe a regular hexagon in the circle O.

Construction and Proof.	
1. Draw any radius OA.	1. Post. 1.
2. With A as a center and AO as a radius, describe an arc intersecting the circle in B.	2. Post. 3.
3. Draw BO and AB.	3. Post. 1.
4. Then $\triangle ABO$ is equilateral.	4. Ax. 1, § 73.
5. $\therefore \angle O = 60°$.	5. § 112.
6. Hence $\overset{\frown}{AB} = 60°$, or $\frac{1}{6}$ of the circle.	6. § 236.
7. With B as a center and a radius $= AB$, describe an arc intersecting the circle in C.	7. Post. 3.
8. Then $\overset{\frown}{BC} = \overset{\frown}{AB} = 60°$.	8. § 199.
9. In like manner, construct arcs CD, DE, EF, and FA, each $= 60°$, and draw their chords.	9. Reasons 7 and 8, Post 1.
10. Then $ABCDEF$ is a regular hexagon inscribed in the circle O.	10. §361. Q.E.F.

365. Cor. 1. *The side of a regular inscribed hexagon equals the radius of a circle.*

366. Cor. 2. *By joining the alternate vertices of a regular inscribed hexagon, an equilateral triangle can be inscribed in a given circle.*

367. Cor. 3. *By bisecting the arcs AB, BC, CD, etc., of the given circle, and drawing chords, a regular polygon of* 12 *sides may be inscribed in a given circle; by repeating the process, regular polygons of* 24, 48, . . . 3×2^n *sides may be inscribed, where n is a positive integer greater than* 1.

368. A **regular pentagon** or **decagon** can be inscribed in a circle by the use of the ruler and compasses, but the consideration of these cases lies beyond the scope of this book.

A regular polygon of seven, nine, or eleven sides cannot be inscribed in a circle by the use of the ruler and compasses.

Proposition IV. Theorem

369. *A circle can be circumscribed about any regular polygon, and a circle can also be inscribed in any regular polygon.*

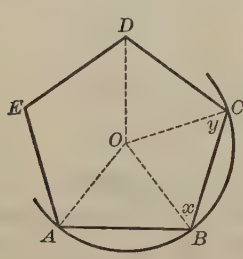


Fig. 1

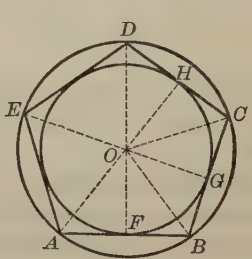


Fig. 2

Given the regular polygon $ABCDE$.

To prove that a $\odot$ can be circumscribed about, or inscribed in, $ABCDE$.

Proof. I. 1. Through A, B, and C (Fig. 1), any three successive vertices of the polygon $ABCDE$, construct a circle, with center O.	1. § 215.
2. Draw the radii OA, OB, OC. Also draw OD.	2. Post. 1.
3. Then, in $\triangle OBC$, $OB=OC$.	3. Why?
4. $\therefore \angle x=\angle y$.	4. Why?
5. But $\angle ABC=\angle BCD$.	5. § 359.
6. Hence $\angle OBA=\angle OCD$.	6. Ax. 3.
7. $AB=CD$.	7. Why?
8. $\therefore \triangle ABO\cong\triangle OCD$.	8. Why?
9. $\therefore OD=OA$.	9. Why?
10. Hence the circle which passes through the vertices A, B, and C will pass through D also.	10. § 181.
11. In like manner, it may be proved that this circle will pass through the vertex E.	11. Reasons 2–10.
12. Hence a circle will be circumscribed about the polygon $ABCDE$.	12. § 227.
II. 1. The sides AB, BC, CD, etc. (Fig. 2), of the given polygon are chords in the circumscribed circle.	1. § 185.
2. Draw OF, OG, etc., ⊥ from O to the sides.	2. § 131.
3. $OF=OG=OH$, etc.	3. Why?
4. With O as a center and OF as a radius, describe a circle.	4. Post. 3.
5. This circle will touch all the sides of the given polygon.	5. §§ 208, 211.
6. Hence a circle will be inscribed in the polygon $ABCDE$.	6. § 228. **Q.E.D.**

Ex. 1. How many degrees in each angle of a regular pentagon?

Ex. 2. Prove that the diagonals of a regular pentagon are equal.

EXERCISES: GROUP 73

In a circle whose radius is one inch, inscribe

Ex. 1. A square. **Ex. 3.** A regular octagon.

Ex. 2. A regular hexagon. **Ex. 4.** An equilateral triangle.

Ex. 5. On a given line as a side, construct a regular hexagon.

Ex. 6. In a circle whose radius is $1\frac{1}{2}''$, inscribe a regular polygon of 12 sides.

Ex. 7. In a circle whose radius is 3 inches, inscribe a square. Find the length of a side of the square. Also find the area of the square.

Ex. 8. In a given circle, the side of an inscribed square is 1.5 in. Find the radius of the circle.

Ex. 9. A log is 14 inches in diameter. Find a side of the largest square piece of lumber that can be cut from the log.

Ex. 10. The diagonals of a regular quadrilateral are perpendicular to each other.

Ex. 11. A square inscribed in a circle is greater than any other inscribed rectangle.

[Sug. Let the inscribed square and any inscribed rectangle with which it is compared have the same diagonal. Show that this diagonal is a diameter, etc.]

Ex. 12. A cooper, in fitting a head to a barrel, adjusts a pair of compasses till, when applied six times in succession in the chine, they will exactly complete the circumference. He then takes the distance between the points of the compasses as the radius of the head of the barrel. Why is this?

370. The **center of a regular polygon** is the common center of the inscribed and circumscribed circles; as the point O in Fig. 2, p. 256.

371. The **radius of a regular polygon** is the radius of the circumscribed circle; as OA in Fig. 2, p. 256.

372. The **apothem** of a regular polygon is the radius of the inscribed circle; as OF in Fig. 2.

373. The **angle at the center of a regular polygon** is the angle between two radii drawn to the extremities of any side; as the angle AOB in both Fig. 1 and Fig. 2.

374. Cor. *The angle at the center of a regular polygon is equal to four right angles divided by the number of sides.*

Hence, if n denotes the number of sides in the polygon, the *angle at the center of a regular polygon equals* $\frac{4 \text{ rt. } \angle s}{n}$.

EXERCISES: GROUP 74

Find the number of degrees in the central angle of

Ex. 1. A regular pentagon.

Ex. 2. A regular hexagon.

Ex. 3. A square.

In a circle whose radius is 10 in., find the apothem and area of

Ex. 4. The inscribed square.

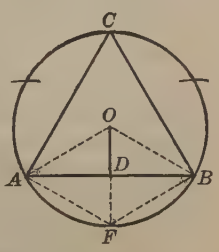
Ex. 5. The regular inscribed hexagon.

In a circle whose radius is b, find the apothem and area of

Ex. 6. The regular inscribed hexagon.

Ex. 7. The inscribed square.

Ex. 8. In a given circle, a side of the inscribed square is 8 in. Find the apothem of this square.



Ex. 9. Prove that in any circle the apothem of the regular inscribed triangle equals half the radius.

[Sug. Produce the apothem OD to meet the circle at F. Then use § 128.]

Ex. 10. Find the apothem and area of an equilateral triangle whose side is 6 in.

Ex. 11. Find the radius of an equilateral triangle whose side is $4\sqrt{3}$.

Ex. 12. Diagonals drawn from a vertex of a regular polygon of n sides divide the angle at the vertex into $(n-2)$ equal parts.

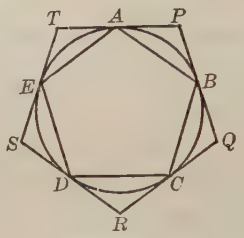
Ex. 13. Using a scale of 1 in. to 20 ft., construct a drawing to represent a regular hexagonal flower bed each side of which is 15 ft. Also find the area of the flower bed.

Proposition V. Theorem

375. *If a circle is divided into equal arcs,*

I. *The chords of these arcs form a regular inscribed polygon;* and

II. *The tangents at the points of division form a regular circumscribed polygon.*



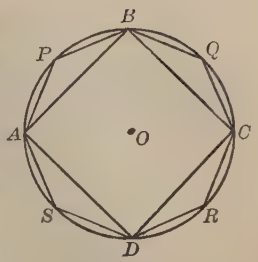
Given the circle ABC divided into the equal arcs AB, BC, CD, etc., the chords AB, BC, etc., and PQ, QR, etc., lines tangent to the circle at B, C, etc.

To prove $ABCDE$ a regular inscribed polygon, and $PQRST$ a regular circumscribed polygon.

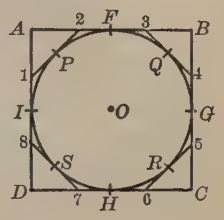
Proof. I. 1. $\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}$, etc.	1. Why?
2. $\therefore AB=BC=CD$, etc.	2. Why?
3. Hence $ABCDE$ is a regular inscribed polygon.	3. §§ 361, 227.
II. 1. In the $\triangle APB$, BQC, CRD, etc., $AB=BC=CD$, etc.	1. Why?
2. $\angle PAB=\angle PBA=\angle QBC=\angle QCB$, etc.	2. §§243, 247.
3. $\triangle APB\cong\triangle BQC\cong\triangle CRD$, etc.	3. § 83.
4. $\therefore \angle P=\angle Q=\angle R$, etc.	4. Why?
5. $AP=PB=BQ=QC=CR$, etc.	5. Corr. sides of congruent $\triangle$ are $=$.
6. $\therefore PQ=QR=RS$, etc.	6. Ax. 4.
7. $\therefore PQRST$ is a regular circumscribed polygon.	7. §§ 359, 228.

Q.E.D

376. Cor. 1. *If the vertices of a regular inscribed polygon are joined with the midpoints of the arcs determined by the sides of the polygon, the joining lines will form a regular inscribed polygon of double the number of sides.*



377. Cor. 2. *If, at the midpoints of the arcs joining the adjacent points of contact of the sides of a regular circumscribed polygon, tangents are drawn, a regular circumscribed polygon of double the number of sides will be formed.*



EXERCISES: GROUP 75

About a circle whose radius is one inch, circumscribe

Ex. 1. An equilateral triangle. **Ex. 3.** A regular hexagon.

Ex. 2. A square. **Ex. 4.** A regular octagon.

For a circle whose radius is 10 in., find the area of

Ex. 5. The circumscribed square.

Ex. 6. The inscribed and circumscribed regular triangles.

Ex. 7. Prove that in a given circle the area of the inscribed regular hexagon is double the area of the inscribed regular triangle.

Ex. 8. In any circle, find the ratio of the area of the inscribed square to that of the circumscribed square.

Ex. 9. Find the ratio of the area of the inscribed regular hexagon to that of the area of the circumscribed regular hexagon.

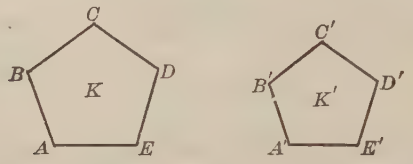
Ex. 10. Construct a six-pointed star (called a hexagram).

Ex. 11. DFH is an equilateral triangle. FH if bisected at P and PH at Q. Prove that the square on DQ is equal to 13 times the square on QH.

Ex. 12. Construct a square whose side is $1\frac{1}{2}$ in. Also a square whose side is 2 in. Then construct a square equal to the difference of these squares.

Proposition VI. Theorem

378. *Regular polygons of the same number of sides are similar.*



Given K and K' two regular polygons, each of n sides.

To prove K and K' similar.

Proof. 1. Each $\angle$ of $K = \frac{2(n-2) \text{ rt. } \measuredangle}{n}$.	1. § 167.
2. Each $\angle$ of $K' = \frac{2(n-2) \text{ rt. } \measuredangle}{n}$.	2. Why?
3. Hence K and K' are mutually equiangular.	3. Ax. 1.
4. Also $AB = BC = CD$, etc.	4. Why?
5. And $A'B' = B'C' = C'D'$, etc.	5. Why?
6. Hence $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.	6. Ax. 5.
7. $\therefore$ K and K' are similar.	7. § 300.
	Q.E.D.

Ex. 1. For a circle whose radius is 8 inches, find the area of the circumscribed regular hexagon.

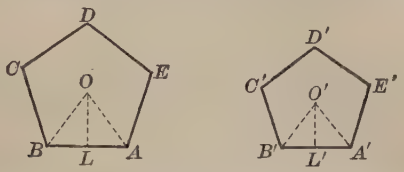
Ex. 2. Prove that in a given circle the side of a regular circumscribed triangle is double the side of the regular inscribed triangle.

Ex. 3. Prove that in any circle the ratio of the area of the inscribed equilateral triangle to the area of the circumscribed equilateral triangle is 1 to 4.

Ex. 4. Prove that the altitude of an inscribed equilateral triangle is to the diameter of the circumscribed circle as 3 is to 4.

Proposition VII. Theorem

379. *The perimeters of two regular polygons of the same number of sides are to each other as the radii of the inscribed circles, or as the radii of the circumscribed circles.*



Given AC and $A'C'$ two regular polygons, each of n sides, with centers O and O', with apothems OL and $O'L'$, and with perimeters denoted by p and p', respectively.

To prove $\frac{p}{p'}=\frac{OL}{O'L'}=\frac{OA}{O'A'}$.

Proof. 1. Polygon $AC\sim$polygon $A'C'$.	1. § 378.
2. $\therefore p : p'=AB : A'B'$.	2. § 315.
3. $\angle BOA=\frac{4 \text{ rt. } \measuredangle}{n}=\angle B'O'A'$.	3. § 374, Ax. 1.
4. $OA=OB$, and $O'A'=O'B'$.	4. § 189.
5. $\triangle OAB\sim\triangle O'A'B'$.	5. § 309.
6. $AB : A'B'=OL : O'L'=OA : O'A'$.	6. §§ 300, 314.
7. $p : p'=OL : O'L'=OA : O'A'$.	7. Ax. 1. Q.E.D.

380. Cor. *The areas of two regular polygons of the same number of sides are to each other as the squares of the radii of the inscribed, or circumscribed, circles.*

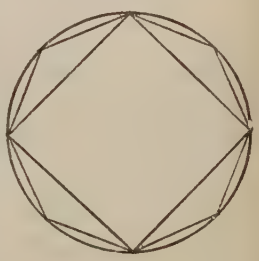
Ex. 1. If the sides of two regular octagons are 3 in. and 6 in., respectively, find the ratio of the perimeters of the octagons. Also the ratio of their areas.

Ex. 2. Find the ratio of the areas of two regular hexagons whose sides are $1\frac{1}{2}$ in. and $2\frac{1}{4}$ in. Also find the ratio of their perimeters Of their apothems.

Ex. 3. The sides of two regular polygons of the same number of sides are 2.4 in. and 3.6 in. The area of the first polygon is 184.32 sq. in. Find the area of the second polygon.

Ex. 4. Two logs are 12 in. and 15 in., respectively, in diameter. Find the ratio between the area of the cross section of the largest square beam that can be cut from the first log and that cut from the second log. Also of octagonal beams thus cut.

381. Deriving properties of circles from those of regular polygons.—By methods which are beyond the scope of this book, it may be shown that, if the number of sides of a regular inscribed or regular circumscribed polygon is repeatedly doubled, the polygons remaining regular, the perimeter of each of these polygons approaches nearer and nearer to the length of the circle: and also the area of each of these polygons approaches nearer and nearer to the area of the circle.



Hence we employ the following principle (the proof of which lies beyond the scope of this book) as a means of deriving the properties of circles from those of regular polygons:

Any property of regular polygons which does not depend on the number of sides of the polygon is true also of circles.

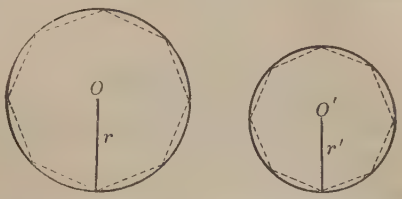
Ex. 1. In a circle whose radius is 10, find the side of the inscribed regular triangle.

Ex. 2. In a circle whose radius is r, show that a side of the regular inscribed triangle is $r\sqrt{3}$.

Ex. 3. In a circle whose radius is 10 in., find the difference between the perimeters of the regular inscribed hexagon and triangle. If the radius of the circle were increased to 20 in., by use of proportion find what the perimeters would become, and then find their difference.

Proposition VIII. Theorem

382. *The circumferences of two circles are to each other as their radii.*



Given the circles O and O' with circumferences denoted by C and C', and radii by r and r', respectively.

To prove $C : C' = r : r'$.

Proof. 1. In the given circles, let regular polygons of the same number of sides be inscribed, and denote their perimeters by p and p'.	1. § 363.
2. $p : p' = r : r'$.	2. § 379.
3. Hence $C : C' = r : r'$.	3. § 381.
	Q.E.D.

383. Cor. *The circumferences of two circles are to each other as their diameters.* For, denote the circumferences by C and C'. Then

$$\frac{C}{C'} = \frac{r}{r'} \text{ (§ 382)} = \frac{2r}{2r'} \text{ (§ 232)} = \frac{d}{d'} \text{ (§ 190)}.$$

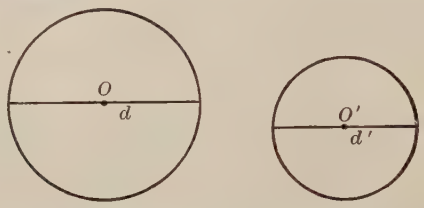
Ex. 1. If the diameter of the earth is approximately 8000 miles and that of the moon is 2000 miles, find the ratio of the lengths of the equators of the two globes.

Ex. 2. In a circle whose radius is 6 in., find the difference between the areas of the inscribed regular hexagon and triangle. Also the difference between the areas of the inscribed square and equilateral triangle.

Ex. 3. On a line $1\frac{1}{4}$ in. long as a chord, construct a segment of a circle which shall contain an angle of 45°. Also a segment containing an angle of 60°. Of 120°.

Proposition IX. Theorem

384. *The ratio of the circumference of any circle to its diameter is constant.*



Given the circle O with its circumference denoted by C and its diameter by d.

To prove that the value of $\frac{C}{d}$ is always the same, no matter what the size of the given circle.

Proof. 1. Construct another fixed circle O' to be used as a standard of reference. Denote its circumference by C' and its diameter by d'.	1. Post. 3.
2. $\frac{C}{C'}=\frac{d}{d'}$.	2. § 383.
3. $\therefore \frac{C}{d}=\frac{C'}{d'}$.	3. § 283.
4. Hence $\frac{C}{d}$ has the same value as $\frac{C'}{d'}$ in the fixed circle; that is, $\frac{C}{d}$ is constant.	4. § 251. Q.E.D.

385. Formulas for the circumference.—If we denote the constant $\frac{C}{d}$ by the symbol π, $\frac{C}{d}=\pi$; hence, $C=\pi d$.

Also, since $d=2r$, $C=\pi(2r)=2\pi r$.

386. Numerical value of π.—By means of methods which are beyond the scope of this book it is found that, when expressed as a number, $\pi=3.141592^{+}$. A convenient approximation to this value is $\pi=\frac{22}{7}$.

387. Formula for the length of an arc of a circle.—$\frac{\text{arc}}{\text{circumference}} = \frac{\text{central } \angle}{360°}$ (§ 235). $\therefore \text{arc} = \frac{\text{central } \angle}{360°} \times 2\pi r.$

$$\therefore \text{arc} = \frac{\text{central } \angle}{180°} \times \pi r.$$

EXERCISES: GROUP 76

Using $\pi = \frac{22}{7}$, find the circumference of the circle in which

Ex. 1. $r = 14$ in. **Ex. 3.** $r = 1\frac{3}{4}$ ft. **Ex. 5.** $d = 3.25$ ft.

Ex. 2. $d = 21$ in. **Ex. 4.** $d = 2\frac{1}{2}$ yd. **Ex. 6.** $r = .0158$ in.

Ex. 7. To how many decimal places is $\frac{22}{7}$ a correct value for π?

Find the radius and diameter of the circle in which the circumference is

Ex. 8. 88 in. **Ex. 10.** $\frac{22}{21}$ in.

Ex. 9. 13.2 in. **Ex. 11.** 18π.

Ex. 12. The diameters of two wheels are 22 in. and 33 in., respectively. Find the ratio of their circumferences.

Ex. 13. The circumferences of two hot-air pipes are 25 in. and 35 in., respectively. Find the ratio of the diameters of the pipes.

In a circle whose diameter is 14 in., find the length of an arc of

Ex. 14. 60°. **Ex. 15.** 75°. **Ex. 16.** 117°. **Ex. 17.** 22° 45′.

Ex. 18. A wheel with 6 cogs is geared to a wheel with 48 cogs. How many revolutions will the smaller wheel make while the larger wheel revolves once?

Ex. 19. In a circle whose radius is 21 in., a certain arc is 33 in. long. How many degrees are there in this arc?

Ex. 20. How many degrees are there in an arc 5.5 ft. long, if the radius of the circle is 14 ft.?

Ex. 21. How many degrees are there in an arc 25 ft. long, if the radius of the circle is 10 ft.?

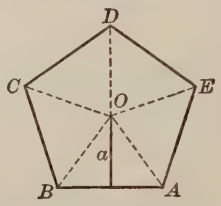
Ex. 22. If the radius of a circle is multiplied by 3, by what number is the circumference multiplied?

Ex. 23. In a circle whose radius is $1\frac{1}{2}$ in., construct a segment which shall contain an angle of 45°. Of 60°.

Ex. 24. Draw two circles whose radii are 1 and $1\frac{1}{2}$ in., respectively, and in each circle construct a segment containing an angle of 120°.

Proposition X. Theorem

388. *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*



Given the regular polygon $ABCDE$ with area denoted by A, perimeter by p, and apothem by a.

To prove $A = \frac{1}{2}ap$.

Proof. 1. Draw the radii OA, OB, OC, etc., dividing the polygon into as many $\triangle$ as the polygon has sides.	1. Post. 1.
2. All of these $\triangle$ have the same altitude, a.	2. § 189.
3. Area of each $\triangle = \frac{1}{2}$ product of its base by a.	3. § 342.
4. Hence, sum of areas of $\triangle = \frac{1}{2}$ (sum of bases of $\triangle) \times a = \frac{1}{2}ap$.	4. Axs. 2, 7.
5. $\therefore A = \frac{1}{2}ap$.	5. Ax. 9.
	Q.E.D.

389. A **sector** of a circle is the figure formed by two radii and the arc intercepted by them.

390. Similar sectors are sectors in different circles which have equal angles at the center.

391. Similar segments are segments in different circles whose arcs subtend equal angles at the center.

Ex. In a given circle construct a segment which is greater than the sector with the same arc.

Proposition XI. Theorem

392. *The area of a circle is equal to half the product of its circumference by its radius.*

Given a ⊙ with circumference denoted by C, radius by r, and area by A.

To prove $A = \frac{1}{2}rC$.

Proof. 1. Circumscribe a regular polygon about the given circle, and denote its perimeter by p, and its area by A'.	1. §§ 364, 375.
2. The apothem of the circumscribed regular polygon is the radius of the given circle, or r.	2. § 372.
3. $A' = \frac{1}{2}rp$.	3. § 388.
4. $\therefore A = \frac{1}{2}rC$.	4. § 381.
	Q.E.D.

393. Formulas for the area of a circle.—Substituting for C its value $2\pi r$ (§ 385),

$$A = \tfrac{1}{2}(2\pi r)r, \text{ or } A = \pi r^2.$$

Again, $r = \frac{1}{2}d$. $\therefore A = \frac{1}{4}\pi d^2$.

394. Cor. 1. *The area of a circle is equal to the square of the radius, multiplied by π; or to one fourth the square of the diameter, multiplied by π.*

395. Cor. 2. *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.* The reason for this is as follows:

Denoting the areas of any two circles by A and A', their radii by r and r', and their diameters by d and d', respectively,

$$\frac{A}{A'}=\frac{\pi r^2}{\pi r'^2}=\frac{r^2}{r'^2};\ \text{also}\ \frac{A}{A'}=\frac{\frac{1}{4}\pi d^2}{\frac{1}{4}\pi d'^2}=\frac{d^2}{d'^2}.$$

396. *The* **area of a sector** *is to the area of the circle as the angle of the sector is to* 360°.

$$\text{Hence}\ \frac{\text{area of sector}}{\pi r^2}=\frac{\text{central}\ \angle}{360°}.$$

$$\text{Or area of sector}=\frac{\text{central}\ \angle}{360°}\times\pi r^2.$$

397. Cor. 3. *The areas of similar sectors are to each other as the squares of their radii.*

EXERCISES: GROUP 77

Using $\pi=\frac{22}{7}$, find the area of a circle in which

Ex. 1. $r=14$ in. **Ex. 3.** $r=.014$ ft. **Ex. 5.** $d=2\frac{3}{4}$ in.

Ex. 2. $d=21$ in. **Ex. 4.** $d=.28$ in. **Ex. 6.** $r=1\frac{1}{2}$ in.

Ex. 7. $r=a$ ft. **Ex. 8.** $d=\frac{1}{4}a$ ft.

Find the area of the circle in which the circumference is

Ex. 9. 176 ft. **Ex. 10.** 1.32 in. **Ex. 11.** $2\frac{2}{11}$ in. **Ex. 12.** 36π.

Ex. 13. The circumference of a hot-air pipe is 22.5 in. Find the area of a cross section of the pipe.

Ex. 14. If the radius of one circle is 10 times as great as the radius of another, find the ratio of the areas of the circles. Also of their circumferences. Of their diameters.

Ex. 15. Assuming that the amount of water flowing through a pipe is proportional to the area of the cross section, a 2-inch pipe will discharge how many times as much water in a given time as a 1-inch pipe?

Ex. 16. The side of a square inscribed in a given circle is 6 in. Find the area of the circle.

Find the radius of a circle of which the area is

Ex. 17. 1386. **Ex. 18.** $16\pi b^2$. **Ex. 19.** 400π. **Ex. 20.** 100 sq. ft.

Ex. 21. In a circle whose radius is 14 in., find the area of a sector of 60°. Of 80°. Of 21°.

Find the area of a circle

Ex. 22. Whose radius is b ft.; $\frac{1}{2}b$ ft.; $2b$ ft.

Ex. 23. Whose radius is $\frac{1}{2}R$; $2R$; $R\sqrt{3}$.

Ex. 24. Whose diameter is $\frac{1}{2}R$; $R\sqrt{2}$; $\frac{1}{2}R\sqrt{2}$.

Ex. 25. In heating a house by a hot-air furnace, the area of the cross section of the cold-air box should equal the sum of the areas of the pipes conducting hot air from the furnace. If a given furnace has three hot-air pipes, each 6″ in diameter, and one pipe 8″ in diameter, and the width of the cold-air box is 15″, how deep should the box be?

Ex. 26. What is the most convenient way of determining the diameter of a hot-air pipe if you have no calipers and the ends of the pipe are not accessible?

Ex. 27. A belt runs over two wheels, one of which has a diameter of 3 ft. and the other of 6 in. If the first wheel is making 120 revolutions per minute, how many is the second wheel making? How many revolutions per minute must the first wheel make in order that the second may make 300 revolutions in the same time?

Assuming (what is not strictly true, owing to friction against the sides of the pipe, etc.) that the rate of flow through a cylindrical pipe is proportional to its area of cross section:

Ex. 28. If a $1\frac{1}{2}$-inch pipe is replaced by a 1-inch pipe, how much is the flow of water decreased?

Ex. 29. A 3-inch pipe is to be replaced by one which will deliver not less than twice as much water per minute. Find, to the nearest quarter of an inch, the diameter of the new pipe.

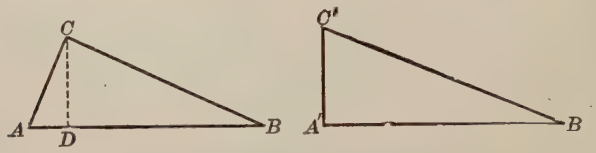
Ex. 30. A city of 40,000 people is barely supplied with water by 12-inch mains from the reservoirs. If these mains are torn out, and 18-inch mains substituted, what future population of the city is allowed for?

Ex. 31. The weight which a round (cylindrical) bar of metal will sustain suspended from its end is proportional to the area of the cross section of the bar. A steel bar 1 in. in diameter will hold up 50,000 lb. What load would be sustained by a bar of the same material $\frac{3}{4}$ in. in diameter? What would be the diameter of a cylindrical bar which is to sustain 150,000 lb.?

MAXIMA AND MINIMA

Proposition XII. Theorem

398. *Of all triangles which have two sides equal, that triangle in which these sides include a right angle is the maximum.*



Given the $\triangle ABC$ and $A'B'C'$ in which $AB=A'B'$, $CA=C'A'$, $\angle A'$ is a rt. $\angle$, and $\angle A$ an oblique $\angle$.

To prove $\triangle A'B'C' > \triangle ABC$.

Proof. 1. Construct $CD \perp AB$.	1. § 131.
2. Then $CD < CA$.	2. Why?
3. $\therefore CD < C'A'$.	3. Ax. 9.
4. $AB = A'B'$.	4. Hyp.
5. $\therefore$ area $\triangle ABC$: area $\triangle A'B'C' = CD : C'A'$.	5. § 345.
6. $\triangle A'B'C' > \triangle ABC$.	6. For $C'A' > CD$. Q.E.D.

399. Maximum plane figure of given perimeter.—By methods which are beyond the scope of this book, it may be shown that, for regular polygons having a given perimeter, the greater the number of sides, the greater the area; and hence that, *of all plane figures having a given perimeter, the circle is the maximum.*

EXERCISES: GROUP 78

Ex. 1. Find the minimum line that can be drawn having an end in each of two given parallel lines.

Ex. 2. Find the length of the longest stick that can be placed on a rectangular table 12×5 ft., and not have an end projecting over a side of the table.

Ex. 3. A grazier has a movable and flexible fence, $\frac{1}{2}$ mile long, with which he incloses his sheep when grazing. Find the difference between the amount of ground inclosed when the fence is arranged in the form of a square and of a circle.

Ex. 4. Find the area of a triangle in which two of the sides are 6 and 12 in., and the included angle is 90°. Find the area of another triangle having two sides of 6 and 12 in., and the included angle 60°.

Ex. 5. How long is the fence about a garden 60×40 ft.? How many square feet are there in the area of the garden? Find also the length of fence and area of a garden 50 ft. square.

Ex. 6. Find the area of an equilateral triangle, a square, a regular hexagon, and a circle, in each of which the perimeter is 1 ft.

Ex. 7. Find the perimeter of an equilateral triangle, a square, and a circle, in each of which the area is 24 sq. in.

SYMMETRY

400. Symmetry of polygons.—Many of the properties of regular figures can be obtained in a simple and expeditious way by the use of the ideas of symmetry.

The principles of symmetry are also useful in treating of such properties of physical bodies as their centers of gravity.

401. An **axis of symmetry** is a line such that, if part of a figure is folded over upon the line as an axis, the part folded over will coincide with the rest of the figure.

EXERCISES: GROUP 79

Ex. 1. How many axes of symmetry has an isosceles triangle? An equilateral triangle?

Ex. 2. How many has a square? A regular pentagon?

Ex. 3. How many has a regular hexagon?

Ex. 4. How many has a regular heptagon? A regular octagon? A regular polygon of n sides? A circle?

Ex. 5. Which capital letters have an axis of symmetry?

402. A center of symmetry for a polygon is a point such that any line drawn through the point and terminated by the polygon is bisected by the point.

EXERCISES: GROUP 80

Ex. 1. Has an equilateral triangle a center of symmetry? Has a square?

Ex. 2. Has a regular pentagon a center of symmetry? Has a regular hexagon?

Ex. 3. In general, which regular polygons have a center of symmetry, and which do not?

Ex. 4. Has a circle a center of symmetry?

Ex. 5. Which is the most symmetrical figure studied thus far?

Ex. 6. A rhombus has how many axes of symmetry? Has it a center of symmetry?

Ex. 7. What axis of symmetry has a quadrilateral which has two pairs of equal adjacent sides? Has such a figure a center of symmetry?

Ex. 8. Prove that a parallelogram is symmetrical with respect to the point of intersection of its diagonals.

Ex. 9. A segment of a circle is symmetrical with respect to what axis?

Ex. 10. Has an isosceles trapezoid a center of symmetry? An axis of symmetry?

Ex. 11. How many axes of symmetry have two equal circles taken as one figure? Have they a center of symmetry?

Ex. 12. What axis of symmetry have any two circles?

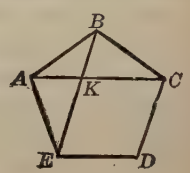
Ex. 13. Which capital letters have a center of symmetry? Small letters?

EXERCISES: GROUP 81

THEOREMS

Ex. 1. If $ABCDE$ is a regular pentagon, the triangles ABK and ABC are similar.

Ex. 2. In the same figure, $BC = KC$.



Ex. 3. Show that the area of a regular inscribed triangle is $\frac{3\sqrt{3}\,r^2}{4}$.

Ex. 4. In a circle whose radius is r, find the difference between the area of an inscribed square and that of an inscribed equilateral triangle.

Ex. 5. In a circle whose radius is r, prove that the side of the inscribed regular octagon is $r\sqrt{2-\sqrt{2}}$.

Ex. 6. In a circle whose radius is r, prove that the side of an inscribed regular polygon of twelve sides is $r\sqrt{2-\sqrt{3}}$.

Ex. 7. From a log whose diameter is a inches a rectangular beam is to be cut so as to have a cross section of the greatest possible area. Find a side of the beam.

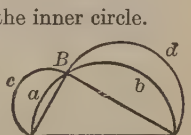
Ex. 8. The diagonals joining the alternate vertices of a regular hexagon form another regular hexagon. Prove this statement; then find the ratio of the areas of the two hexagons.

Ex. 9. If squares are erected on the outside of the sides of a regular hexagon, the lines joining their exterior vertices form a regular dodecagon. Prove this statement and find the area of the dodecagon in terms of b, a side of the hexagon.

Ex. 10. The area of a circle equals four times the area of a circle described on its radius as a diameter.

Ex. 11. The area of a circular ring equals the area of a circle whose diameter is the chord of the outer circle tangent to the inner circle.

Ex. 12. Given $AaBbC$, AcB, BdC, semicircles; prove that the sum of the two crescents $AcBa$ and $BdCb$ equals the area of the right triangle ABC.



Ex. 13. Two right triangles with the same diameter as hypotenuse are inscribed in a circle. One of these right triangles is isosceles and the other is not. Prove that the isosceles triangle has a greater area than the other.

Ex. 14. Each side of a regular hexagon is trisected and a circle is passed through all the points of division. If a side of the hexagon is 6 in., find the area of the circle.

Ex. 15. The side of a square is 8 in. With two opposite vertices as centers and with radii of 8 in., circles are described. Find to the nearest hundredth of a square inch the area common to the two circles.

Ex. 16. A circle is described on one side of an equilateral triangle as a diameter. If a side of the triangle is 6 in., find the area of that part of the triangle that lies without the circle.

EXERCISES: GROUP 82

Construction Problems

Ex. 1. Construct a circumference equal in length to the sum of two given circumferences.

Ex. 2. Construct a circumference equal in length to the difference of two given circumferences.

Ex. 3. Construct a circle whose area is equal to the sum of the areas of two given circles.

Ex. 4. Construct a circle whose area is equal to the difference of the areas of two given circles.

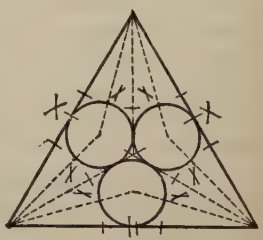
Ex. 5. It is desired to plant 8 trees so that they shall be equidistant from a given point, and also so that each two successive trees shall be 20 ft. apart. Draw a diagram to some definite scale, showing the relative position of the trees.

Ex. 6. Bisect the area of a given circle by drawing another circle concentric with the given one.

Ex. 7. Inscribe a circle in a given sector.

Ex. 8. Inscribe a square in a given segment of a circle.

Ex. 9. In a given equilateral triangle, inscribe three equal circles, each of which touches the other two circles and a side of the triangle.



Ex. 10. In a given circle, inscribe three equal circles which shall touch each other and the given circle.

Ex. 11. Inscribe a regular octagon in a given square.

Ex. 12. Construct a circle whose area shall be equal to the area of a given semicircle.

EXERCISES: GROUP 83

Practical Applications

Ex. 1. A half-mile running track is to have equal semicircular ends and parallel straight sides. The extreme length of the rectangle

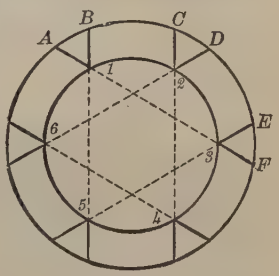
together with the semicircular ends is to be 1000 ft. Find the width of the rectangle.

[Sug. Denote the length of the radius of the semicircular ends by x and that of one of the parallel side straight tracks by y, and obtain a pair of simultaneous equations.]

Ex. 2. If, in laying a track, a rail 10 ft. long is bent through an arc of 5° 10′, what is the radius of the curve?

Ex. 3. A given revolving wheel is liable to fly to pieces if the speed of its outer rim exceeds 5000 ft. per minute. If the diameter of the wheel is 27 in., how many revolutions can the wheel safely make per minute?

Ex. 4. Construct a circular basket by the method indicated in the adjoining diagram. Thus, after drawing two concentric circles, locate the vertices of 1, 2, 3, etc. of the regular hexagon inscribed in the smaller circle. Draw the straight lines as indicated. Cut out the notches $A1B$, $C2D$, $E3F$, etc. Fold up the parts of the outer circular ring which are left, so that B coincides with A, D with C, etc.



Ex. 5. If the center of symmetry of a flat, homogeneous object is its center of mass, find the center of mass of a square. Of a rectangle. Regular hexagon. Circle.

Ex. 6. If a box has a square end, subject to the same pressure at all points, at what single point on the end must a supporting pressure be applied?

Ex. 7. The cross section of a cylinder is a circle. The weight-supporting strength of a horizontal solid cylindrical beam of given material and length varies as the area of the cross section times its radius.

Compare the weight-supporting power of two solid horizontal iron cylindrical beams of the same length and quality of iron, the radii being 3 in. and 6 in. respectively. (Point out and use the short way of getting the desired result.)

Ex. 8. The cross section of a hollow cylinder (*i.e.* of a tube) is a circular ring. Denote the outside radius of the tube by R and the inside radius by r. Then it may be shown that the weight-supporting

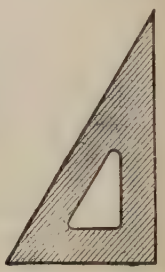
power of a hollow cylindrical tube of given length and material varies as the area of the cross section (*i.e.* of the ring) times

$$\frac{R^2+r^2}{R}.$$

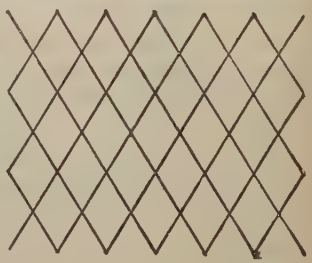
If $R=4$ in. and $r=3$ in., compare the weight-supporting power of the tube with that of a solid cylindrical beam of the same length and the same cross-sectional area.

In general, a cylindrical tube is stronger than a solid cylindrical beam of the same length and containing the same amount of material.

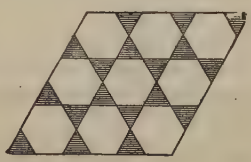
Hence, in a framework, as in that of an aëroplane, where the maximum strength must be obtained from a given amount of material, the metallic rods and posts are all tubular. In like manner, bamboo rods, since they are hollow, are used instead of solid wooden rods in some constructions. For the same reason, the bones of flying birds, and many bones in men and animals, are hollow and not solid.



Ex. 9. Adjoining is a picture of the drawing instrument called a 60-degree triangle. By use of this instrument (or by constructing equilateral triangles and constructing lines parallel to their sides) form several sheets of paper ruled similarly to squared paper except that the lines on them intersect at angles of 60° instead of 90°, thus:

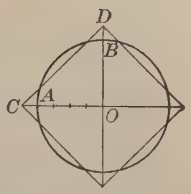


Ex. 10. By the aid of paper ruled as in Ex. 9, construct the following designs, which illustrate the way in which a mosaic or pavement may be formed of regular polygons.

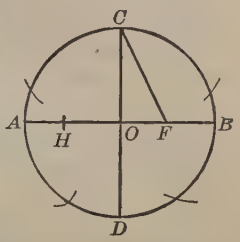


Ex. 11. To construct a square which shall be approximately equivalent to a given circle O, divide the radius OA into four equal

parts, produce each end of two perpendicular diameters a distance equal to one fourth of the radius, and connect the extremities of the lines thus formed. Show that taking the square thus formed as equivalent to the circle is the same as taking $\pi = 3\frac{1}{8}$. Also find the per cent of error in taking this square as equivalent to the circle.



Ex. 12. A short way to construct a regular inscribed pentagon and also a five-pointed star (or pentagram) is as follows: Draw a circle O and two diameters AB and CD at right angles. Bisect the radius OB at F, and with F as a center and FC as a radius describe an arc cutting AO at H. Then CH is the length of a side of the regular inscribed pentagon. By joining the alternate vertices of this pentagon, the pentagram may be formed.



Ex. 13. Construct a square. Take each vertex of the square as a center and one half a side of the square as a radius and outside of the square describe arcs which meet. Erase the square and you have a quatrefoil. By drawing other circles and arcs of circles, elaborate the quatrefoil into an ornamental design. (See the figure for Ex. 11, p. 177.)

Ex. 14. In like manner, construct a cinquefoil by use of a regular pentagon, and develop it into an ornamental design.

Ex. 15. Treat a regular hexagon in the same way.

EXERCISES: GROUP 84

Review of Book Five

Ex. 1. Construct a regular polygon of three sides and on it mark its center, radius, and apothem.

Ex. 2. The measurement of the area of a circle can be reduced to the measurement of the length of what single straight line? Can it be reduced to the measurement of any other single straight line? To the measurement of a single curved line?

Ex. 3. What is the efficiency value of Prop. XI?

Ex. 4. Of Prop. IX? Of Prop. X?

Ex. 5. State some of the ways in which circles aid in dealing with regular polygons.

Ex. 6. State some of the ways in which regular polygons aid in dealing with circles.

Ex. 7. When are two regular polygons similar?

Ex. 8. The circumferences of two circles are C and C', respectively. In terms of C and C', give the ratio of the diameters of these circles. Of their radii. Of their areas.

Ex. 9. How does the side of a regular inscribed hexagon compare with the radius in length? How does the apothem of the inscribed equilateral triangle compare with the radius of the circle?

Ex. 10. Name five different regular polygons which you can inscribe in a circle by use of the ruler and compasses. Name five which you cannot thus inscribe in a circle.

Ex. 11. If a and a' are the sides of two regular polygons of the same number of sides, give the ratio of the perimeters of the polygons. Of their radii. Of their apothems. Of their areas.

Give a list of the properties proved or stated in Book Five concerning

Ex. 12. A single regular polygon.

Ex. 13. Two regular polygons of the same number of sides.

Ex. 14. A single circle.

Ex. 15. Two or more circles taken together.

Ex. 16. Give an account of the history of the successive approximations which have been made to the value of π. (See p. 287.)

HISTORY OF GEOMETRY

403. Origin of geometry as a science.—The beginnings of geometry as a science are found in Egypt, dating back at least three thousand years before Christ. Herodotus says that geometry, as known in Egypt, grew out of the need of remeasuring pieces of land which had been in part washed away by the Nile floods, in order to make an equitable readjustment of the taxes on the land.

The substance of the Egyptian geometry is found in an old papyrus roll, now in the British Museum. This roll is, in effect, a mathematical treatise written by a scribe named Ahmes at least as early as 1700 B.C., and is, the writer states, a copy of a work dating, say, 3000 B. C.

404. Epochs in the development of geometry.—From Egypt a knowledge of geometry was transferred to Greece, whence it spread to other countries. Hence we have the following principal epochs in the development of geometry:

1. **Egyptian:** 3000 B.C.–1500 B.C.
2. **Greek:** 600 B.C.–100 B.C.
3. **Hindu:** 500 A.D.–1100 A.D.
4. **Arab:** 800 A.D.–1200 A.D.
5. **European:** 1200 A.D.–

In the year 1120 A.D., Athelard, an English monk, visited Cordova, in Spain, in the disguise of a Mohammedan student, and procured a copy of Euclid in the Arabic language. This book he brought back to central Europe, where it was translated into Latin and became the basis of all geometric study in Europe till the year 1453, when, owing to the capture of Constantinople by the Turks, copies of the works of the Greek mathematicians in the original Greek were scattered through Europe.

HISTORY OF GEOMETRICAL METHODS

405. Rhetorical methods.—By "rhetorical methods" in the presentation of geometric truths, is meant the use of definitions, axioms, theorems, geometric figures, the representation of geometric magnitudes by the use of letters, the arrangement of material in Books, etc. The **Egyptians** had none of these, their geometric knowledge being recorded only in the shape of the solutions of certain numerical examples, from which the rules used must be inferred.

Thales (Greece, 600 B.C.) first made an enunciation of an abstract property of a geometric figure. He had a crude idea of the geometric theorem.

Pythagoras (Italy, 525 B.C.) introduced formal definitions into geometry, though some of those used by him were not very accurate. For instance, his definition of a point is "unity having position." Pythagoras also arranged the leading propositions known to him in something like logical order.

Hippocrates (Athens, 420 B.C.) was the first systematically to denote a point by a capital letter, and a segment of a line by two capital letters (as the line AB) as is done at present. He also wrote the first textbook on geometry.

Plato (Athens, 380 B.C.) made definitions, axioms, and postulates the beginning and basis of geometry.

To **Euclid** (Alexandria, 280 B.C.) is due the division of geometry into Books, the formal enunciation of theorems, the particular enunciation, the formal construction, proof, and conclusion, in presenting a proposition. He also introduced the use of the corollary.

Using these methods of presenting geometric truths, Euclid wrote a textbook of geometry in thirteen books, which was the standard textbook on this subject for nearly two thousand years.

The use of the symbols $\triangle$, $\square$, $\parallel$, etc., in geometric proofs originated in the United States in recent years.

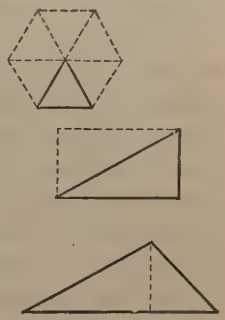
406. Logical methods.—The **Egyptians** used no formal methods of proof. They probably obtained their few crude geometric processes as the result of experiment.

The **Hindus** also used no formal proof. One of their writers on geometry merely states a theorem, draws a figure, and says "Behold!"

The use of logical methods of geometric proof is due to the **Greeks.** The early Greek geometricians used *experimental methods* at times, in order to obtain geometric truths. For instance, they determined that the angles at the base of an isosceles triangle are equal, by folding half of the triangle over on the altitude as an axis and observing that the angles mentioned coincided as a fact, but without showing that they *must* coincide.

Pythagoras (525 B.C.) was the first to establish geometric truths by systematic *deduction*, but his methods were sometimes faulty. For instance, he believed that the converse of a proposition is necessarily true.

Hippocrates (420 B.C.) used correct and *rigorous deduction* in geometric proofs. He also introduced specific varieties of such deduction, such as the method of *reducing* one proposition to another, and the *reductio ad absurdum* (§ 180).



The methods of deduction used by the Greeks, however, were defective in their lack of generality. For instance, it was often thought necessary to have a separate proof of a theorem for each different kind of figure to which the theorem applied. Thus the theorem that the sum of the angles of a triangle equals two right angles was proved,

(1) for the equilateral triangle by use of the regular hexagon;

(2) for the right triangle by the use of a rectangle;

(3) for other triangles by dividing the triangle into two right triangles.

The Greeks appeared to fear that a general proof might be vitiated if it were applied to a figure in any way special or peculiar.

Plato (380 B.C.) introduced the method of proof by *analysis*, that is, by taking a proposition as true and working from it back to known truths. (See § 174.)

To **Eudoxus** (380 B.C.) is virtually due proof by the method of *limits*, though his method, known as the method of exhaustions, is crude and cumbersome.

Apollonius (Alexandria, 225 B.C.) used *projections*, *transversals*, etc., which, in modern times, have developed into the subject of projective geometry.

407. Mechanical methods.—The **Greeks**, in demonstrating a geometrical theorem, usually drew the figure employed in a bed of sand. This method had certain advantages, but was not adapted to demonstration before a large audience.

At the time when geometry was being developed in Greece, the interest in the subject was very general. There was scarcely a town but had its lectures on the subject. The news of the discovery of a new theorem spread from town to town, and the theorem was redemonstrated in the sand of each marketplace.

The Greek treatises, however, were written on vellum or papyrus by the use of the reed, or *calamus*, and ink.

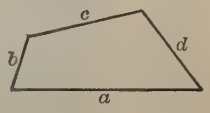
In **Roman times,** and in the **Middle Ages,** geometrical figures were drawn in wax smeared on wooden boards, called tablets. They were drawn by the use of the *stylus*, a metal stick which was pointed at one end for making marks and broad at the other end for erasing marks. These wax tablets were still in use in Shakespeare's time. (See *Hamlet*, Act I, Sc. 5, l. 107.) The blackboard and crayon are **modern** inventions, their use having developed within the last one hundred years.

The **Greeks** invented many kinds of drawing instruments for tracing various curves. It was due to the influence of Plato (380 B.C.) that, in constructing geometric figures, the use of only the ruler and compasses is permitted.

HISTORY OF GEOMETRIC TRUTHS; PLANE GEOMETRY

408. Rectilinear figures.—The **Egyptians** measured the area of any four-sided field by multiplying half the sum of one pair of opposite sides by half the sum of the other pair; which was equivalent to using the formula,

$$\text{area} = \frac{a+c}{2} \times \frac{b+d}{2}.$$



This, of course, gives a correct result for the rectangle and the square, but gives too great a result for other quadrilaterals, as the trapezoid. Joseph, of the Book of Genesis, in buying the fields of the Egyptians for Pharaoh in time of famine by the use of this formula, frequently paid for a larger field than he obtained.

The Egyptians had a special fondness for geometrical constructions, probably growing out of their work as temple builders. A class of workers existed among them called "rope-stretchers," whose business was the marking out of the foundations of buildings. These men knew how to bisect an angle and how to construct a right angle. The latter was probably done by a method essentially the same as forming a right triangle whose sides are three, four, and five units of length. Ahmes, in his treatise, has various constructions of the isosceles trapezoid from different data.

Thales (600 B.C.) enunciated the following theorems:

If two straight lines intersect, the opposite or vertical angles are equal;

The angles at the base of an isosceles triangle are equal;

Two triangles are congruent if two sides and the included angle of one are equal to two sides and the included angle of the other;

The sum of the angles of a triangle equals two right angles;

Two mutually equiangular triangles are similar.

Thales used the last of these theorems to measure the height of the great pyramid; he measured the length of the shadow cast by the pyramid and also measured the length of the shadow of a post of known height at the same time and made a proportion between these quantities.

Pythagoras (525 B.C.) and his followers discovered correct formulas for the areas of the principal rectilinear figures, and also discovered the theorems that the areas of similar polygons are as the squares of their corresponding sides, and that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. The latter is called the Pythagorean theorem. They also discovered how to construct a square equal in area to a given parallelogram, and to divide a given line in mean and extreme ratio.

To **Eudoxus** (380 B.C.) we owe the general theory of proportion in geometry, and the treatment of incommensurable quantities by the method of exhaustions. By the use of these, he obtained such theorems as that the areas of two circles are to each other as the squares of their radii, or of their diameters.

In the writings of **Hero** (Alexandria, 125 B.C.) we first find the formula for the area of a triangle in terms of its sides, $A = \sqrt{s(s-a)(s-b)(s-c)}$. Hero also was the first to place land-surveying on a scientific basis.

It is a curious fact that Hero, at the same time, gives an incorrect formula for the area of a triangle, viz. $A = \frac{1}{2}a(b+c)$, this formula being apparently derived from Egyptian sources.

The **Romans**, although they excelled in engineering, apparently did not appreciate the value of the Greek geom-

THE PYRAMIDS OF EGYPT

By measuring shadows and using proportion, Thales found the heights of the pyramids. Can you explain how he did it? (*Ewing Galloway*)

THE DEATH OF ARCHIMEDES

Though the great Greek mathematician and inventor, Archimedes, unlike modern scientists, thought his inventions were valueless and beneath the dignity of pure science, yet by their use his native city of Syracuse was able to defy Roman conquest for three years. Besides inventing a number of war machines, legend says he used the burning glass to fire the Roman fleet. In the general massacre which came with the final downfall of the city, Archimedes was killed by Roman soldiers while drawing mathematical figures in the sand. A seventeenth century artist here shows the old man calmly drawing his figures on the earth, as was customary among mathematicians in those days, and too absorbed in his theorems to heed the approach of the soldiers.

etry. Even after they became acquainted with it, they continued to use antiquated and inaccurate formulas for areas, some being of obscure origin. Thus, they used the Egyptian formula for the area of a quadrilateral, $A=\frac{a+b}{2}\times\frac{c+d}{2}$. They determined the area of an equilateral triangle whose side is a by different formulas, all incorrect. as $A=\frac{13a^2}{30}$, $A=\frac{1}{2}(a^2+a)$, and $A=\frac{1}{2}a^2$.

409. The circle.—Thales enunciated the theorem that every diameter bisects a circle, and proved the theorem that an angle inscribed in a semicircle is a right angle.

To **Hippocrates** (420 B.C.) is due the discovery of nearly all the other principal properties of the circle given in this book.

The **Egyptians** regarded the area of the circle as equivalent to $\frac{64}{81}$ of the diameter squared, making $\pi=3.1604$.

The **Jews** and the **Babylonians** treated π as equal to 3.

Archimedes (Sicily, 250 B.C.), by the use of inscribed and circumscribed regular polygons, showed that the true value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$; that is, between 3.14285 and 3.1408.

The **Hindu** writers assign various values to π, as 3, $3\frac{1}{8}$, $\sqrt{10}$, and **Aryabhatta** (530 A.D.) gives the correct approximation, 3.1416. The Hindus used the formula $\sqrt{2-\sqrt{4-\overline{AB}^2}}$ in computing the numerical value of π.

Within recent times, the value of π has been computed to 707 decimal places.

The use of the symbol π for the ratio of the circumference of a circle to the diameter was established in mathematics by **Euler** (Germany, 1750).

410. Cancellation.—In numerical work in geometry, as elsewhere, the labor of computations may frequently be economized. Those methods of abbreviating work which are particularly serviceable in the ordinary numerical applications of geometry, may be briefly indicated as follows:

To simplify numerical work by cancellation, *group together as a whole all the numerical processes of a given problem, and make all possible cancellations, before proceeding to a final numerical reduction.*

Ex. Find the ratio of the area of a rectangle, whose base and altitude are 42 and 24 inches, to the area of a trapezoid, whose bases are 21 and 35 and altitude 12 inches.

By §§ 336, 349,

$$\frac{\text{area of rectangle}}{\text{area of trapezoid}} = \frac{42\times 24}{6(21+35)} = \frac{\overset{8}{\cancel{42}}\times\overset{3}{\cancel{24}}}{\underset{8}{\cancel{6}}\times\cancel{56}} = 3, \textit{ Ratio.}$$

411. Use of radicals and of π.—Where radicals enter in the course of the solution of a numerical problem, it frequently saves labor *not to extract the root of the radical till the final answer is to be obtained.*

Ex. 1. Find the area of a circle circumscribed about a square whose side is 8.

The diagonal of the square must be $8\sqrt{2}$ (§ 354).

$\therefore$ the radius of $\odot = 4\sqrt{2}$.

$\therefore$ by § 393, area of $\odot = \pi(4\sqrt{2})^2 = 32\pi = 100.6$, *Area.*

Similarly in the use of π, it frequently saves labor *not to substitute its numerical value for π till late in the process of solution.*

Ex. 2. Find the radius of a circle whose area is equal to the sum of the areas of two circles whose radii are 6 and 8 inches, respectively.

Denote the radius of the required circle by x.

Then, by § 393, $\pi x^2 = 36\pi + 64\pi$.

$\therefore \pi x^2 = 100\pi$.

$\therefore x^2 = 100$,

$\therefore x = 10$, *Radius.*

412. Computation of a numerical result to the nearest final figure.—In computing to the nearest hundredth the numerical value of an expression containing a surd, pupils frequently make the mistake of computing the root involved to two decimal places only.

Ex. Find to the nearest hundredth the value of $15\sqrt{6}$. If we compute the square root of 6 to only two decimal places, we obtain

$$15\sqrt{6} = 15(2.44+) = 36.60+.$$

But if we compute the square root of 6 to five decimal places,

$$15\sqrt{6} = 15(2.44948+) = 36.74220+$$
$$= 36.74+. \quad \textit{Ans.}$$

Hence, in general, in evaluating a radical expression, the larger the coefficient of the radical, the larger should be the number of decimal places to which the root is extracted.

413. Rationalizing a denominator.—If a numerical result contains a surd in the denominator, labor is saved, and the accuracy of the result is increased, by rationalizing the denominator before computing the root involved.

Ex. Find to the nearest hundredth of an inch the side of an equilateral triangle whose area is 10 sq. in.

If x denotes the side of the given triangle, we have

$$\frac{x^2\sqrt{3}}{4} = 10, \quad \text{or} \quad x^2 = \frac{40}{\sqrt{3}}.$$

Rationalizing the denominator, $x^2 = \frac{40\sqrt{3}}{3} = \frac{40(1.73205+)}{3}$.

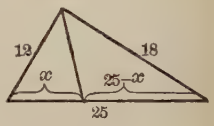
Hence $x = 4.805+$, or $x = 4.81-$. *Ans.*

Let the pupil compute the value of $\frac{40}{\sqrt{3}}$ without rationalizing the denominator (*i.e.*, by dividing 40 by 1.73205+), and compare the amount of work in this process with that in the one followed above.

414. Use of x, y, etc., as symbols for unknown quantities. —In some cases *a numerical computation is greatly facilitated by the use of a specific symbol for an unknown quantity.*

Ex. 1. In a triangle whose sides are 12, 18, and 25, find the segments of the side 25 made by the bisector of the angle opposite.

Denote the required segments of side 25 by x and $25-x$.



Then $12 : 18 = x : 25-x$ (§ 297).

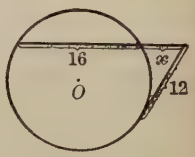
$\therefore 18x = 12(25-x)$ (§ 277).

$\therefore x = 10$

And $25-x = 15$ } *Segments.*

Ex. 2. From an external point a tangent 12 in. long is drawn to a given circle; also a secant whose internal segment is 16 in. Find to the nearest hundredth of an inch the length of the external segment of the secant.

Denote the length of the external segment by x.



Then by § 323, $x(x+16) = (12)^2$.

Or $x^2+16x = 144$.

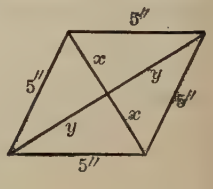
Solving this quadratic equation,

$x = 8 \pm 14.422+$.

$x = 6.42+$. *Segment.*

Ex. 3. The area of a given rhombus is 24 sq. in., and the perimeter of the rhombus is 20 in. Find the diagonals.

Denote the diagonals by $2x$ and $2y$.



Then by §§ 128, 354, $x^2+y^2 = 25$,

and by § 347, $xy = 12$.

Hence $x = 4, 3$

$y = 3, 4$ }

$\therefore$ 8 in., 6 in. *Diagonals.*

415. Limitations of numerical computations.—Owing to the limitations of human eyesight and of the instruments used in making measurements, no measurement can be accurate beyond the fifth or sixth figure; and in ordinary

work, such as is done by a carpenter, measurements are not accurate beyond the third figure. As all numerical applications of geometry are based on practical measurements, *it is not necessary to carry arithmetical work beyond the fifth or sixth significant digit.*

Other methods of facilitating numerical computations, as by the **use of logarithms** or the **slide rule,** are beyond the scope of this book.

In solving numerical examples in this book, compute answers containing decimals to the nearest hundredth, unless otherwise stated.

EXERCISES: GROUP 85

Determine the value of each of the following to the nearest hundredth.

Ex. 1. $7.6263+$.
Ex. 2. $17.9344+$.
Ex. 3. $5.7982+$.
Ex. 4. $3.9964+$.
Ex. 5. $4.0135+$.
Ex. 6. $23.805+$.
Ex. 7. $21.003+$.
Ex. 8. $17.608-$.
Ex. 9. $3.795-$.
Ex. 10. $0.127-$.
Ex. 11. $\frac{3.2715}{4}$.
Ex. 12. $\frac{15-6\frac{1}{8}}{2}$.
Ex. 13. $12-5.6134+$.
Ex. 14. $20\sqrt{3}$.
Ex. 15. $18\sqrt{7}$.
Ex. 16. $3\sqrt{5}$.
Ex. 17. $5+6\sqrt{2}$.
Ex. 18. $28-7\sqrt{5}$
Ex. 19. $13-\frac{1}{2}\sqrt{6}$.
Ex. 20. $8+\frac{3\sqrt{5}}{4}$.
Ex. 21. $\frac{21-2\sqrt{7}}{5}$.

Determine to the nearest hundredth the value of the following, after first rationalizing each denominator:

Ex. 22. $\frac{3}{\sqrt{2}}$.
Ex. 23. $\frac{7}{2\sqrt{3}}$.
Ex. 24. $\frac{3\sqrt{7}}{5\sqrt{5}}$.
Ex. 25. $\frac{12}{\sqrt{300}}$.
Ex. 26. $\frac{5\sqrt{7}-1}{\sqrt{7}+2}$.
Ex. 27. $\frac{1+\sqrt{2}}{2-\sqrt{3}}$.

Ex. 28. Solve in the shortest way $\pi x^2=\pi(225)+\pi(64)$.

Ex. 29. Find to the nearest hundredth the value of x in $x(x-15)=95$.

Ex. 30. Find to the nearest hundredth the values of x and y in $x^2+y^2=17$, $xy=5$.

REVIEW EXERCISES

EXERCISES: GROUP 86

Lines

Ex. 1. A tree 90 ft. high is broken off 40 ft. from the ground. How far from the foot of the tree will the top strike?

Ex. 2. Find the diagonals of a rectangle whose sides are 5 and 12.

Ex. 3. If the base of an isosceles triangle is 8 and a leg is 5, find the altitude.

Ex. 4. One side of a rhombus is 17 and one diagonal is 30. Find the other diagonal.

Ex. 5. In a circle whose radius is 5, find the length of the longest and shortest chords through a point at a distance 3 from the center.

Ex. 6. In a circle whose radius is 25 in., find the distance from the center to a chord 48 in. long.

Ex. 7. If a chord 12 in. long is 5 in. from the center of a circle, how far from the center of the same circle is a chord 10 in. long?

Ex. 8. A ladder 40 ft. long reaches a window 20 ft. high on one side of a street and, if turned on its foot, reaches a window 30 ft. high on the other side. How wide is the street?

Ex. 9. If one leg of a right triangle is 10 and the hypotenuse is twice the other leg, find the hypotenuse.

Ex. 10. Find the altitude of an equilateral triangle whose side is 6.

Ex. 11. Find the side of an equilateral triangle whose altitude is 8.

Ex. 12. Find the side of a square whose diagonal is 15.

Ex. 13. One leg of a right triangle is 3, and the sum of the hypotenuse and the other leg is 9. Find the sides.

Ex. 14. The radii of two circles are 1 in. and 6 in., and their centers are 13 in. apart. Find the length of the common external tangent.

Ex. 15. The sides of a triangle are 12, 18, and 20. Find the segments of the side 20, made by the bisector of the angle opposite.

Ex. 16. If the legs of a right triangle are 6 and 8, find the hypotenuse, the altitude on the hypotenuse, and the segments of the hypotenuse made by the altitude.

Ex. 17. Two sides of a triangle are 17 and 16, and the altitude on the third side is 15. Find the third side.

Ex. 18. The hypotenuse of a right triangle is 10, and the altitude on the hypotenuse is 4. Find the segments of the hypotenuse and the legs.

Ex. 19. A bicycle wheel 28 in. in diameter makes, in an afternoon, 3000 revolutions. How many miles does the bicycle travel?

Ex. 20. What is the diameter of a wheel which makes 1400 revolutions in going 8800 yd.?

Ex. 21. If the diameter of a circle is 20, find the length of an arc of 60°. Also of 83°.

Ex. 22. If the length of an arc is 14 and the radius is 6, find the number of degrees in the arc.

Ex. 23. If the arc of a quadrant is 1 ft. in length, find the diameter.

Ex. 24. Two concentric circles are 88 in. and 132 in. in length, respectively. Find the width of the circular ring between them.

Ex. 25. If the year is taken as $365\frac{1}{4}$ days, and the earth's orbit is taken as a circle whose radius is 93,250,000 miles, find the velocity of the earth in its orbit per second.

Find the radius and circumference of a circle circumscribed about

Ex. 26. A square whose side is 5.

Ex. 27. A rectangle whose sides are 12 and 5.

Ex. 28. Find the central angle of a sector whose perimeter equals one half the circumference.

Ex. 29. Find the radius of a circle whose circumference equals the perimeter of a square whose diagonal is 10.

Ex. 30. Two intersecting chords of a circle are 11 in. and 14 in. and the segments of the first chord are 8 in. and 3 in. Find the segments of the second chord.

Ex. 31. In a circle whose radius is 12 in., a chord 16 in. long is passed through a point 9 in. from the center. Find the segments of the chord.

Ex. 32. Two secants drawn from a point to a circle are 24 in. and 27 in. long. If the external segment of the first is 6 in., find the external segment of the second.

Ex. 33. From a given point a secant, whose external and internal segments are 9 and 16, is drawn to a circle. Find the length of the tangent drawn from the same point to the circle.

Ex. 34. From a given point a tangent 24 in. long is drawn to a circle whose radius is 18 in. Find the distance of the point from the center.

Ex. 35. If a diameter 60 in. long is divided into 5 equal parts by chords perpendicular to it, find the length of the chords.

Ex. 36. If the greatest distance at which a mountain 3 miles high is visible at sea is 150 miles, what is the diameter of the earth?

Ex. 37. If the earth is a sphere of radius 4000 mi., how far will the light of a lighthouse 100 ft. high be visible at sea?

Ex. 38. In a triangle whose base is 14 and altitude 12, a line is drawn parallel to the base and at a distance 2 from the base. Find the length of the line thus drawn.

Ex. 39. The upper and lower bases of a trapezoid are 12 and 20 and the altitude is 8. If the legs are produced till they meet, find the altitude of each of the two triangles thus formed.

Ex. 40. If the upper and lower bases of a trapezoid are b and b', and the altitude is h, find the altitude of each of the triangles formed by producing the legs.

Ex. 41. If the sides of a triangle are 6, 7, and 8, compute the length of the altitude on 8.

Ex. 42. Also the length of the median on the same side.

Ex. 43. Also the length of the bisector of the angle opposite the side 8.

Ex. 44. Find the three medians, the three bisectors, and the three altitudes of a triangle whose sides are 13, 14, 15.

EXERCISES: GROUP 87

Areas

Ex. 1. Find to three decimal places the area in acres of a triangular field whose base is 300 ft. and altitude 200 ft.

Ex. 2. Find the area of a triangle whose sides are 10, 17, and 21.

Ex. 3. Find the area in acres of a triangular field each of whose sides is 10 chains.

Find the area of

Ex. 4. An isosceles triangle whose base is 16, and each of whose legs is 34.

Ex. 5. An equilateral triangle whose altitude is 8.

Ex. 6. An isosceles right triangle whose hypotenuse is 12.

Ex. 7. A right triangle in which the hypotenuse is 41 and one leg is 9.

Ex. 8. The area of an equilateral triangle is $4\sqrt{3}$. Find a side.

Ex. 9. Find the number of boards, each 4 yd. long and 6 in. wide, which are necessary to cover a floor 48×24 ft.

Ex. 10. How many persons can stand in a room 15×9 ft., if each person requires 27×18 in.?

Ex. 11. The baseball diamond is a square each side of which is 90 ft. What fraction of an acre is its area?

Ex. 12. Find the area of a rhombus one of whose sides is 17, and one of whose diagonals is 30.

Ex. 13. The bases of an isosceles trapezoid are 20 and 36 and the legs are 17. Find the area.

Ex. 14. The base of a triangle is 20 and the altitude 18. Find the length of a line parallel to the base which cuts off a trapezoid whose area is 80 sq. ft.

[Sug. Denote the altitude of the trapezoid by $18-x$ and find its upper base by similar triangles.]

Ex. 15. The perimeter of a polygon, circumscribed about a circle whose radius is 20, is 340. Find the area of the polygon.

Ex. 16. The area of a rectangle is 144 and the base is three times the altitude. Find the dimensions.

Ex. 17. Find a side of a regular hexagon whose area is 200 sq. in.

Ex. 18. Find the area of a circle whose circumference is p.

Ex. 19. Find the radius of a circle whose area equals the sum of the areas of two circles whose radii are 9 in. and 40 in.

Ex. 20. Find the radius of a circle whose area equals the sum of the areas of three circles whose radii are 20, 28, 29.

Ex. 21. In a circle of radius 50, find the area of a sector of 80°.

Ex. 22. Also of a segment of 60°; of a segment of 300°; of a segment of 240°.

Ex. 23. In a circle whose radius is 7, the area of a sector is 45 sq. ft. Find the number of degrees in its angle.

Ex. 24. In a circle whose radius is 10, find the sum of the segments formed by an inscribed square.

Ex. 25. A circular mill-pond, $\frac{1}{2}$ mi. in diameter, contains a circular island, 100 yd. in diameter. Find the water surface of the pond in acres.

Ex. 26. Two tangents to a circle, whose radius is 15, include an angle of 60°. Find the area included between the tangents and the radii to the points of contact.

Ex. 27. Find the length of the tether by which a cow must be tied, in order that she may graze over exactly one acre.

Ex. 28. Three equal circles touch each other externally. Show that the area included between them is $R^2\left(\sqrt{3}-\frac{\pi}{2}\right)$.

Ex. 29. In a triangle whose base is 24 in. and altitude is 18 in., the altitude is bisected by a line parallel to the base. Find the area of the triangle cut off.

Ex. 30. In the triangle of Ex. 29, what part of the altitude must be cut off in order that the area of the triangle may be bisected?

Ex. 31. In a circle whose diameter is 30 in., what are the diameters of concentric circles which divide the area into three equivalent parts?

EXERCISES: GROUP 88

General Numerical Exercises in Plane Geometry

Ex. 1. The leg of an isosceles triangle is 10 and the base is 16. Find the altitude and the area.

Ex. 2. Find the area of a triangle whose sides are 25, 39, 40. Also find the radius of a circle equal in area to this triangle.

Ex. 3. Find the area of a regular hexagon inscribed in a circle whose radius is 2.

Ex. 4. The sides of a triangle are 7, 8, and 9 inches. Find the sides of a similar triangle of four times the area. Also, of twice the area.

Ex. 5. If the sides of a triangle are 12, 16, and 21, what are the segments of the side 21 made by the bisector of the angle opposite?

Ex. 6. The sides of a quadrilateral in order are 5, 5, 4, 3, and the first two of these sides contain an angle of 60°. Find the area.

Ex. 7. Find the diameter of a wheel which, in a mile, makes 480 revolutions.

Ex. 8. The area of a trapezoid is 112 and the two bases are 12 and 16. Find the altitude.

Find the radius of a circle equal in area to

Ex. 9. A square whose side is 10.

Ex. 10. An equilateral triangle whose side is 12.

Ex. 11. A trapezoid whose bases are 16 and 18 and altitude 9.

Ex. 12. A circle, a square, and an equilateral triangle each have a perimeter of 12 yd. Find the area of each figure.

Ex. 13. In a circle whose area is 400, the area of a sector is 125. Find the angle of the sector.

Ex. 14. How many acres are included within a half-mile running track, if the track is in the shape of a rectangle twice as long as it is wide?

Ex. 15. In a square whose side is 6 in., find the area of the inscribed and of the circumscribed circles.

Ex. 16. One leg of a right triangle is 12, and the difference between the hypotenuse and the other leg is 8. Find the area.

Ex. 17. Find the area of an isosceles right triangle whose hypotenuse is 20 ft.

Ex. 18. In a circle whose diameter is 20, a chord is passed through a point at a distance 6 from the center, perpendicular to the diameter through that point. Find the length of this chord, and of the chords drawn from its extremities to the ends of the diameter.

Ex. 19. If three arcs, each of 60° and having 10 for a radius, are each concave to the other two arcs, find the area included by them.

Ex. 20. A square piece of land and a circular piece each contain 1 acre. How many more feet of fence does one require than the other?

Ex. 21. If the base of a triangle is doubled and the altitude remains unchanged, how is the area affected? If the altitude is doubled and the base remains unchanged? If both the base and the altitude are doubled?

Ex. 22. Find the side of an equilateral triangle equal in area to a circle whose diameter is 10.

Ex. 23. The area of a rhombus is 156 sq. in. and one side is 1 ft. 1 in. Find the diagonals.

EXERCISES: GROUP 89

Exercises Involving the Metric System

Ex. 1. Find the area of a triangle of which the base is 16 dm. and the altitude 80 cm.

Ex. 2. Find the area of a triangle whose sides are 6 m., 70 dm., 800 cm.

Ex. 3. Find the area in square meters of a circle whose radius is 14 dm.

Ex. 4. If the hypotenuse of a right triangle is 17 dm. and one leg is 150 cm., find the other leg and the area.

Ex. 5. If the circumference of a circle is 1 m., find the area of the circle in square decimeters.

Ex. 6. Find the area in hectares, and also in acres, of a circle whose radius is 100 m.

Ex. 7. If the diagonal of a rectangle is 35 dm. and one side is 800 mm., find the area in square meters, and also in square inches.

Ex. 8. Find the area of a trapezoid whose bases are 600 cm. and 2 m., and whose altitude is 80 dm.

Ex. 9. If a rectangular field is 700 dm. long and 200 m. wide, find its area in hectares and in acres.

Ex. 10. In a given circle two chords, whose lengths are 15 dm. and 13 dm., intersect. If the segments of the first chord are 12 dm. and 3 dm., find the segments of the second chord.

Ex. 11. Find in decimeters the radius of a circle equal in area to a square whose side is 1 ft. 6 in.

Ex. 12. Find in feet the diameter of a wheel which, in going 10 kilometers, makes 5000 revolutions.

EXERCISES: GROUP 90

Review Exercises in Plane Geometry

Ex. 1. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplementary.

Ex. 2. If a diagonal of a quadrilateral bisects two of its angles, the diagonal bisects the quadrilateral.

Ex. 3. Through a given point draw a secant at a given distance from the center of a given circle.

Ex. 4. The bisectors of one angle of a triangle and of an exterior angle at another vertex form an angle which is equal to one half the third angle of the triangle.

Ex. 5. The side of a square is 18 in. Find the circumference of the inscribed and circumscribed circles.

Ex. 6. The quadrilateral $ADBC$ is inscribed in a circle. The diagonals AB and DC intersect in the point F. Arc $AD=112°$, arc $AC=108°$, $\angle AFC=74°$. Find all the other angles of the figure.

Ex. 7. Find the locus of the center of a circle which touches two given equal circles.

Ex. 8. The line joining the midpoints of two radii of a given circle is perpendicular to the line bisecting their angle.

Ex. 9. If a quadrilateral is inscribed in a circle and its diagonals are drawn, how many pairs of similar triangles are formed?

Ex. 10. A rectangular piece of cardboard is $12''\times14''$. From this is cut an isosceles trapezoid with bases of $14''$ and $8''$ and sides of $5''$; and also a square with sides of $8''$. What per cent of the cardboard is wasted? Draw to scale a diagram illustrating this problem.

Ex. 11. Draw a square $ABCD$. On the diagonal AC take the point E so that $AE=AB$, and draw through E a perpendicular to AE cutting BC in F. Prove $BF=EC$.

Ex. 12. In a circle whose radius is 12 cm., find the length of the tangent drawn from a point at a distance 240 mm. from the center.

Ex. 13. If two non-adjacent sides of a regular pentagon are produced, find the angle of their intersection.

Ex. 14. In the parallelogram $ABCD$, points are taken on the diagonals such that $AP=BQ=CR=DS$. Show that $PQRS$ is a parallelogram.

Ex. 15. A chord 6 in. long is at the distance 4 in. from the center of a circle. In the same circle, find the distance from the center of a chord 8 in. long.

Ex. 16. If B is a point on a circle whose center is O, PA a tangent at any point P, meeting OB produced at A, and PD perpendicular to OB, then PB bisects the angle APD.

Ex. 17. Construct a parallelogram, given a side, a diagonal, and half of the other diagonal.

Ex. 18. Find in inches the sides of an isosceles right triangle whose area is 1 sq. yd.

Ex. 19. Find the locus of the vertices of all triangles on a given base and having a given area.

Ex. 20. If, on the sides AC and BC of the triangle ABC, the squares AD and BF are constructed, AF and DB are equal.

Ex. 21. Prove that in a quadrilateral the angle between the bisectors of two adjacent angles is one half the sum of the other two angles.

Ex. 22. Let A and B be two fixed points on a circle, and P and Q the extremities of a variable diameter of the same circle. Find the locus of the point of intersection of the straight lines AP and BQ.

Ex. 23. Find the area of a circle in which the length of an arc of $22\frac{1}{2}°$ is $\frac{3\pi}{2}$ ft.

Ex. 24. The sum of the areas of two circles is 20 sq. yd., and the difference of their areas is 15 sq. yd. Find their radii.

Ex. 25. Find the altitude of a rhombus of area 100 and side 15.

Ex. 26. If perpendiculars are drawn to a given line from the vertices of a parallelogram, the sum of the perpendiculars from two opposite vertices equals the sum of the other two perpendiculars.

Ex. 27. Two equal circles overlap in such a way that the center of each circle lies upon the other circle. In terms of π and r, find an expression for the area common to the two circles.

Ex. 28. A given circle has an area of 80 sq. ft. In this circle, find the length of an arc of 80°.

Ex. 29. The area of a given triangle is K and the sides of the triangle are a, b, c. In terms of a, b, and c, find the radius of the circle inscribed in the triangle.

Ex. 30. In a triangle ABC, a point P is taken on AC so that the ratio of AP to PC is $\frac{2}{3}$. Through P lines are drawn parallel to AB and BC within the triangle. What is the ratio of the area of the resulting parallelogram to the area of the triangle ABC?

Ex. 31. In a square $ABCD$, E is the midpoint of CD. Construct and describe the locus of P, the midpoint of a line drawn parallel to BE and terminated by the sides of the square.

Ex. 32. If E is the intersection of the diagonals AC and BD of a quadrilateral, and the triangle ADE is equal in area to the triangle BEC, then the lines AB and CD are parallel.

Ex. 33. A 2-inch steam pipe conveying steam from the boiler to the radiators in a school building is found to supply only two thirds the needed amount of steam. If available sizes of pipe are each an exact number of inches in diameter, what is the diameter of the smallest pipe that will convey the needed amount?

Ex. 34. FH is perpendicular to the line AB and is bisected by AB at O. P is a point on the same side of AB as F, and PQ is perpendicular to AB and meets AB at Q. PH intersects AB at R. Prove the angles FRO and PRQ equal.

Ex. 35. $ABCD$ is a rectangle. CP bisects the angle BCA and AP bisects the angle BAC. PR is perpendicular to AC, meets CD produced at R, and intersects AD at O. Prove $PR = CR$, and $AO = PO$.

Ex. 36. Given a parallelogram, with sides 2 in. and 3 in., one of whose angles is 60°. A circle of radius 1 in. rolls around outside the parallelogram, always touching it. Draw accurately the locus of the center of the circle, and compute the length of this locus. Also compute the area included by the locus.

REVIEW AND DIAGNOSTIC TESTS

Note to the Teacher: Tests I–XIII of the following series are designed for review work throughout the year and may be used at the intervals noted. Enough material can be selected from each test for a full class period, or, for the longer tests, several class periods may be used. An attempt has been made to test a maximum of subject matter and the ability to apply it, with a minimum of writing or copying.

Tests XIV and XV are given as samples of good new-type tests, which may be used as final reviews of a full year's work. Each of these tests consists of two parts: Part I, a test of the pupil's knowledge of the tools of geometry—definitions, theorems, constructions, and the like—made up of short questions involving one-step or two-step inferences; and Part II, a test of the pupil's ability to use these tools in the solution of originals, both theorematic and numerical. Part I should occupy one hour, while Part II should occupy two hours.

Test I

(To follow the completion of Proposition III, Book I)

A. Statements to Be Completed

Directions: In this and similar tests, number your paper and then write beside each number the word or words needed to fill each blank.

1. Two angles whose sum equals a right angle are ——.
2. —— angles are angles whose sum is equal to a straight angle.
3. The sum of a right angle and an acute angle is an —— angle.
4. One third of a straight angle is an —— angle.
5. The complement of $\angle A$ is less than the complement of $\angle B$; therefore $\angle A$ is —— than $\angle B$.
6. A —— angle equals its supplement.
7. An —— angle is greater than its supplement.
8. The sum of the two adjacent angles formed by one straight line meeting another straight line equals a —— angle.

9. A —— is a straight line that makes a right angle with a given straight line.

10. —— angles are angles which have a common vertex and a common side between them.

11. Three points not in the same straight line determine —— straight lines.

B. True-False Statements

Directions: In this and similar tests, number your paper. Then beside each number write "True" if you think the statement is true, or "False" if you think it is false. Remember that, in geometry, a statement must be considered false unless it holds true for every possible case.

1. If two supplementary angles are equal, they are right angles.

2. An angle may be equal to its complement.

3. All adjacent angles are equal.

4. The size of an angle can be changed by changing the lengths of the sides.

5. If two angles are each 38°, they are equal.

6. Vertical angles cannot be supplementary.

7. The sum of two sides of a triangle is less than the third side.

8. When one complementary angle is twice the other, the smaller angle contains 60°.

9. The supplement of an angle is always greater than the angle.

10. Two triangles are congruent if two sides and an angle of one are equal to the corresponding parts of the other.

11. An acute angle is one half of an obtuse angle.

12. The bisectors of two supplementary adjacent angles form a right angle.

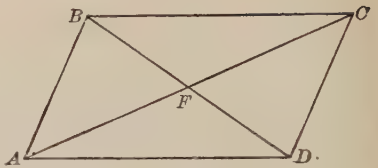
13. An equilateral triangle is also isosceles.

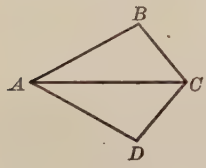
14. The difference between the complement and the supplement of an angle is always 90°.

15. An angle can have only one bisector.

C. Problems and Exercises

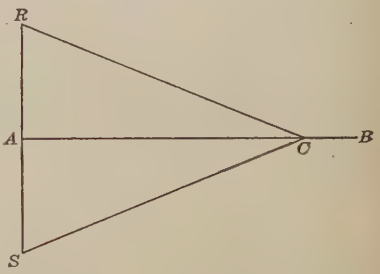
1. Given $\angle BCA = \angle CAD$, $BC = AD$.
To prove $\triangle ABC \cong \triangle ADC$.

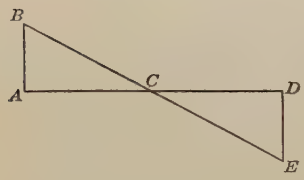




2. Given AC the bisector of $\angle BCD$, $BC = DC$.
To prove $\triangle ABC \cong \triangle ADC$.

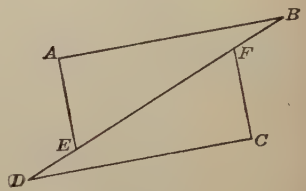
3. Given AB perpendicular bisector of RS, C any point in AB.
To prove $RC = SC$.

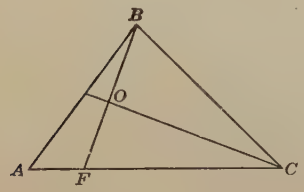




4. Given C the midpoint of AD, $AB \perp AD$, $DE \perp AD$.
To prove $AB = DE$.

5. Given $AB = DC$, $BF = DE$, $\angle D = \angle B$.
To prove $\triangle ABE \cong \triangle DFC$.





6. Given $\triangle ABC$, CO bisector of $\angle BCA$ and $\perp BF$.
To prove $BC = FC$.

Test II

(To follow the completion of Proposition IX, Book I)

A. Statements to Be Completed

1. The bisector of the vertex angle of an isosceles triangle is perpendicular to the ——.
2. The line from the vertex to the base of an isosceles triangle is perpendicular to the base if it —— the base.
3. A point on the bisector of an angle is equally distant from the —— of the angle.
4. The perpendicular bisector of the base of an isosceles triangle passes through the —— of the triangle.
5. A bisector divides a figure into —— —— parts.
6. If two adjacent angles have their exterior —— in the same straight line, they are ——.
7. The figure formed by producing two straight lines from a point is an ——.
8. Only one straight line can be drawn between two ——.
9. If one straight line meets another straight line and makes the adjacent angles equal, the lines are ——.
10. If one of two vertical angles is 125°, the other angle is ——.
11. Geometric figures which —— are ——.

B. True-False Statements

1. A shadow is a surface.
2. Vertical angles and adjacent angles are equal.
3. A figure with two sides equal is isosceles.
4. *Bisect* means *to cut.*
5. A straight angle is equal to two right angles.
6. If two isosceles triangles have the same base, the line drawn through their vertices and cutting the base forms four right triangles.
7. A triangle is a polygon.
8. Two angles which together are equal to a right angle are complementary.
9. An obtuse angle is less than its supplement.

10. In the $\triangle ABC$, $\angle A = 25°$, $\angle B = 35°$, and the exterior angle at $C = 30°$.

11. Only one line can be drawn bisecting a given line.

12. A straight line can be drawn through any three points.

13. Two adjacent angles are straight angles.

14. Two angles are complementary if their sum is 90°.

15. An angle of 36° equals one fourth of its supplement.

C. Problems and Exercises

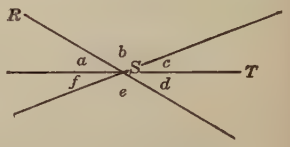
1. Given angle A, angle B, and side AB. Construct a triangle ABC.

2. Construct a right triangle (angle B is the right angle) whose base AB is 2 in., and whose angle $A = 75°$.

3. (a) Bisect angle A of the triangle drawn for Question 2.
   (b) Construct a perpendicular at the midpoint of the side AB of the triangle drawn in Question 1.

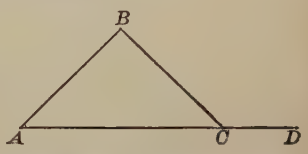
4. (a) Find the angle whose complement is one third of its supplement.
   (b) Find the angle which exceeds its complement by 10°.

5. (a) If $\angle a = 30°$ and $\angle e = 130°$, find $\angle RST$. Give reason.
   (b) If $\angle d = 40°$ and $\angle c =$ one half $\angle d$, find $\angle b$. Give reason.



6. Give proof (omit reasons): If two sides of a triangle are equal, the angles opposite these sides are equal.

7. Given $\angle BCD$ an exterior angle of triangle ABC.
   To prove $\angle BCD$ is greater than $\angle B$. (Omit reasons.)



8. What two previous theorems are used in proving two triangles are congruent if three sides of one are equal to three sides of the other?

TEST III

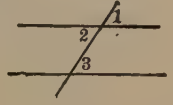
(To follow the completion of Proposition XVIII, Book I)

A. Statements to Be Completed

1. In a triangle, there can be only one —— angle or one —— angle.
2. If the sides of one angle are parallel respectively to the sides of another angle, the angles are —— or ——.
3. In $\triangle ABC$, $AB = AC$, $\angle A = 35°$. Then $\angle B =$ ——.
4. If one base angle of an isosceles triangle is double the vertex angle, each base angle contains —— degrees.
5. Each base angle of an isosceles triangle is —— an exterior angle at the vertex.
6. In $\triangle RLT$, $\angle R = 40°$, $\angle L = 60°$, $\angle T =$ ——.
7. Angles adjacent to the hypotenuse of a right triangle are ——.
8. The sum of two angles of a triangle is equal to a straight angle —— the third angle.
9. The angle formed by the bisectors of the acute angles of a right triangle contains —— degrees.
10. Interior angles on the same side of the transversal cutting two parallel lines are ——.
11. Alternate interior angles of —— lines are ——.
12. The bisectors of an interior and an exterior angle at a vertex of a triangle form a —— angle.
13. Two equilateral triangles are —— if a side of one —— a side of the other.
14. If an angle of an isosceles triangle is 60°, the triangle is ——.

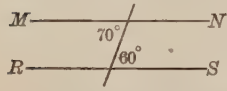
B. True-False Statements

1. Alternate interior angles are equal.
2. When two angles are equal, each angle contains 45°.
3. A right angle can contain two 45° angles.
4. If $\angle 1 = \angle 2$, the lines are parallel.



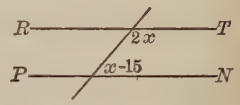
5. In the $\triangle ABC$, $AC = BC$, and BF bisects $\angle B$. If $\angle BFC = 105°$, then $\angle C = 30°$.

6. In the $\triangle RLT$, $\angle R = 63°$, $\angle L = 71°$. Therefore the exterior angle at $T = 135°$.



7. MN is parallel to RS.

8. RT is parallel to PN. Therefore $x = 55°$.



9. Two straight lines perpendicular to the same straight line are parallel.

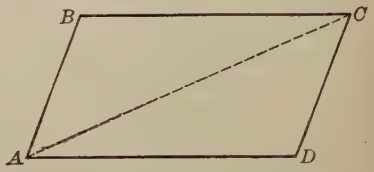
10. If two angles of one triangle equal the corresponding angles of another, the remaining angles of the triangles are equal.

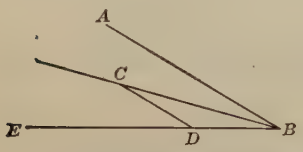
11. Corresponding angles of parallel lines are equal.

C. Problems and Exercises

1. Prove: Two lines cut by a transversal are parallel if the alternate interior angles are equal.

2. **Given** $BC \parallel AD$
$DC \parallel AB$.
To prove $BC = AD$, $AB = DC$, $\angle B = \angle D$, $\angle BAD = \angle BCD$.

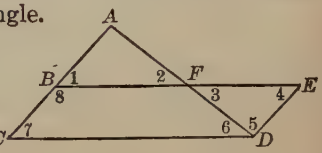


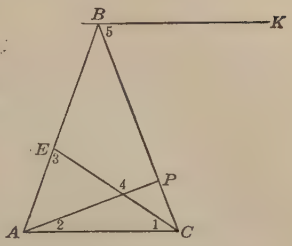


3. **Given** $\angle ABE$ bisected by CB, $\triangle CDB$ isosceles.
To prove $AB \parallel CD$.

4. Prove, omitting reasons: The sum of the interior angles of a triangle equals one straight angle.

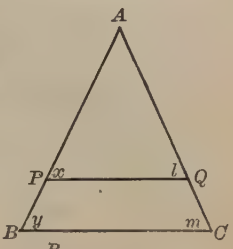
5. Given $BE \parallel CD$, $DE \parallel AC$.
Then $\angle 1 = \angle ? = \angle ?$
$\angle 2 = \angle ? = \angle ?$
$\angle 1$ is supplement of $\angle ?$





6. Given $AB = CB$, $AP \perp BC$ EC bisects $\angle BCA$, $BK \parallel AC$.

If $\angle ABC = 40°$, how many degrees are there in $\angle BAC$, $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$?



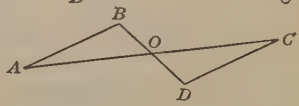
7. Do not give proofs. Answer questions only.

(a) Given $AB = AC$, $PQ \parallel BC$.

In proving $\angle x = \angle l$, what theorem concerning parallel lines is used?

(b) Given AB equal and parallel to DC.

In proving that AC and BD bisect each other at O, what theorem concerning parallel lines is used?



TEST IV

(To follow the completion of Proposition XXV, Book I)

A. Review Questions

1. How many lines can bisect a given angle?

2. If two angles are equal, are their sides always parallel?

3. How many degrees are there in each acute angle of an isosceles right triangle?

4. In $\triangle ABC$,

(a) Can $\angle A = \angle B = \angle C$?

(b) Can $\angle A = 2 \angle B = 3 \angle C$?

(c) Can $\angle A = \angle B + \angle C$?

(d) If $\angle A$ is increased 15° and $\angle B$ is decreased 20°, does the value of $\angle C$ change?

(e) Are the values of the angles of a triangle dependent upon one another?

5. If an exterior angle of a triangle is 100°, one of the opposite interior angles must have a value less than how many degrees?

6. In $\triangle ABC$, $\angle A = 50°$, $\angle B = 75°$. What is the value of the exterior angle at C?

7. Can there be more than two angles which are adjacent?

8. How long is the bisector of an angle?

9. May the shape of a triangle be changed without changing the length of the sides?

10. Where are all the points equidistant from the extremities of a given line located?

11. Can a triangle have sides 6, 9, and 16?

12. How long is a line segment?

13. If two angles are supplementary, what is their sum?

14. Can a triangle be constructed whose sides are $3\frac{1}{2}''$, $4\frac{1}{2}''$, and $8''$?

15. How many perpendiculars can be drawn from a point to a line?

16. Can two adjacent angles be straight angles?

17. Which sides of a right triangle are perpendicular?

18. How many acute angles can a right triangle have?

19. How many points determine a straight line?

20. All the points equidistant from the sides of an angle are on what line?

21. In the $\triangle ABC$, $\angle A = 30°$, $\angle B = 30°$. Is the triangle isosceles?

22. How many lines are determined by four points not in the same straight line?

23. If an angle of 110° is bisected, what is the value of each part?

24. If two angles of a triangle are equal, are the sides opposite the equal angles equal?

25. Is it true that not more than two points can be equidistant from the ends of a line?

26. Can an angle be divided into 3 equal parts by means of ruler and compass?

B. Problems and Exercises

Prove 1–4:

1. Every point in the perpendicular bisector of a line-segment is equidistant from the ends of the segment.

2. If a point is equidistant from the sides of an angle, it lies in the bisector of that angle.

3. If two angles of a triangle are equal, the sides opposite these angles are equal.

4. Two right triangles are congruent if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.

5. Construct triangle ABC, given $AB = 3$ in., angle $A = 60°$, angle $B = 45°$. In this triangle construct the median to AC; the bisector of angle C; a parallel to BC through A.

Test V

(To follow the completion of Book I)

A. Selection Test

Directions: Number your paper. After the numbers write the letters of those words or groups of words which, either together or alone, make the statements true. Thus, for the first exercise, write 1. (a), (c).

1. Every (a) square, (b) rectangle, (c) rhombus, (d) parallelogram, has diagonals perpendicular to each other.

2. Every (a) parallelogram, (b) quadrilateral, (c) isosceles trapezoid, (d) equilateral triangle, (e) isosceles triangle, has just two equal sides.

3. A (a) square, (b) rectangle, (c) trapezoid, (d) rhombus, is an equilateral parallelogram.

4. The diagonals of every (a) square, (b) rhombus, (c) parallelogram, (d) rectangle, are equal.

5. The diagonals of every (a) trapezoid, (b) square, (c) parallelogram, (d) rhombus, bisect the angles.

6. In all parallelograms, (a) the diagonals are equal, (b) the opposite angles are supplementary, (c) the diagonals bisect each other, (d) the opposite sides are equal.

7. If the angles of a triangle have the ratio 1 : 2 : 3, (a) two of its angles are equal, (b) one angle is a right angle, (c) one acute angle is double the other, (d) the two acute angles are equal.

8. All equal complementary angles are (a) adjacent, (b) of the same size, (c) the supplements of each other, (d) 45° angles, (e) vertical angles.

9. In a triangle, an exterior angle (a) equals the adjacent interior angle, (b) equals the sum of the opposite interior angles, (c) is supplementary to the adjacent interior angle, (d) equals a right angle.

10. If a transversal crosses two parallel lines, (a) the interior angles on the same side of the transversal are supplementary, (b) the angles formed are equal, (c) one of the interior angles on one side of the transversal is acute and the other is obtuse, (d) the sum of the exterior angles is 180°.

11. A line bisects one side of a triangle if (a) it bisects a second side, (b) it bisects an angle, (c) it is an altitude, (d) it is parallel to a third side.

12. Two angles are equal if (a) they are angles of a parallelogram, (b) they have parallel sides, (c) they are corresponding parts of congruent figures, (d) they are obtuse angles.

13. Two consecutive angles of all (a) triangles, (b) squares, (c) rectangles, (d) trapezoids, are supplementary.

14. The figure formed by joining the midpoints of the halves of the diagonals of any parallelogram (a) is a parallelogram, (b) is equilateral, (c) has a perimeter equal to one half the perimeter of the given parallelogram, (d) is a rectangle.

15. The opposite angles of all (a) parallelograms, (b) squares, (c) rectangles, (d) trapezoids, are supplementary.

16. Two angles are always equal if (a) their sides are respectively perpendicular, (b) they are alternate interior angles of parallel lines, (c) they are opposite angles in a rectangle, (d) they are acute angles.

17. Every (a) square, (b) rhombus, (c) parallelogram, (d) rectangle, (e) trapezoid, has four equal angles.

18. An equilateral triangle is most like a (a) rhombus, (b) rectangle, (c) trapezoid, (d) square.

B. Statements to Be Completed

1. The difference between two sides of a triangle is —— than the third side.

2. A quadrilateral is a parallelogram if the opposite sides are ——.

3. In $\triangle ABC$, if side AB is greater than side AC, and if $\angle B = 60°$, then $\angle A$ is —— than $\angle B$.

4. If a line 4″ long joins the midpoints of two sides of a triangle, the length of the third side is —— inches.

5. One angle of a right triangle is 60°. If the hypotenuse is 12, the shortest side is ——.

6. In a parallelogram, any two consecutive angles are ——.

7. If the diagonals of a quadrilateral bisect each other, the figure is a ——.

8. An exterior angle of a triangle is 132°. If one opposite interior angle is 66°, the triangle is ——.

9. A transversal crosses two parallel lines making a pair of alternate interior angles $3x + 70°$ and $x + 120°$ respectively. There are —— degrees in each acute angle formed.

10. If one angle of a triangle is twice the first, and the third angle is three times the first, the angles contain ——, ——, and —— degrees respectively.

11. A quadrilateral with two sides equal and the other two parallel may be either a —— or a ——.

12. If one angle of a parallelogram is 69°, the other angles are ——, ——, and ——.

13. A line which is parallel to one side of a triangle and —— a second side, —— the third side.

14. The adjacent sides of a parallelogram are 4 in. and 6 in. Hence the length of each diagonal must be less than —— in. and more than —— in.

15. If the bases of a trapezoid are 6 in. and 13 in. respectively, the length of the median is —— in.

16. If one base angle of an isosceles triangle is double the vertex angle, there are ——, ——, and —— degrees respectively in each of the three angles.

17. An interior angle of a regular decagon contains —— degrees.

18. The line which joins the midpoints of the legs of a trapezoid is called the ——.

19. The diagonals of a —— are equal.

20. If the hypotenuse of a right triangle is double the shorter leg, the angles contain ——, ——, and —— degrees respectively.

21. The bisectors of the vertical angles formed by two intersecting lines are ——.

22. The vertices of all isosceles triangles on the same base lie in the —— of the base.

C. Problems and Exercises

Directions: Answer 1 or 2 and the 6 others.

Prove 1–3:

1. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.

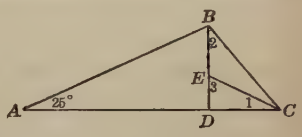
2. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

3. Give *statements* only. (Omit reasons.)

(a) Two triangles are congruent if three sides of one are equal respectively to three sides of the other.

(b) Every point equidistant from the ends of a segment lies in the perpendicular bisector of the segment.

4. Given triangle ABC with BD an altitude and CE bisecting $\angle BCA$, $\angle A = 25°$, $\angle 1 = 27°$.

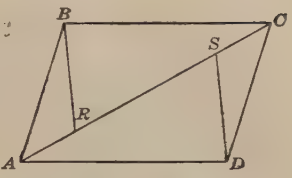


(a) How many degrees are there in $\angle BCD$, $\angle 2$, $\angle 3$, $\angle ABC$? (Omit reasons. Show all numerical work.)

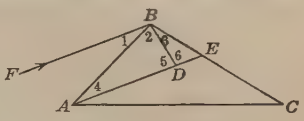
(b) How many degrees are in an angle whose supplement is 4 times its complement?

5. Prove: If, from any point on a leg of an isosceles triangle, a line is drawn parallel to the base, the new triangle formed is isosceles.

6. Given parallelogram $ABCD$, AC a diagonal, and $AR = CS$.
To prove $BR \parallel DS$.



7. Given triangle ABC, AE any line from A to BC, and $FB \parallel AE$, $AB = AD$.
Answer "True" or "False" for each of the following statements, giving a reason for each answer:



(a) $\angle 4 = \angle 1$.
(b) $\angle 1 + \angle 2 + \angle 5 = 180°$.
(c) $\angle 3 > \angle 5$.
(d) $\angle 2 = \angle 5$.
(e) $AB + BD < AD$.
(f) $\angle 3 + \angle 6 = 180°$.

8. (a) Construct an isosceles right triangle of which one leg is equal to $1\frac{1}{2}$ in.
(b) Construct the altitude from the vertex angle.

Test VI

(To follow the completion of Proposition XVI, Book II)

A. Statements to Be Completed

1. The longest chord of a circle is the ——.
2. From any point on a circle —— chords may be drawn.
3. A straight line may intersect a circle in —— points.
4. A straight line joining the extremities of an arc is called a ——.
5. Tangents drawn at the extremities of a diameter are ——.

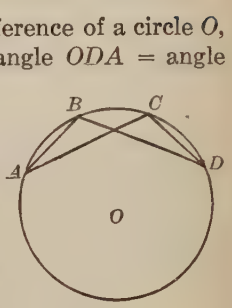
B. True-False Statements

1. Any line perpendicular to a chord bisects the chord.
2. If a central angle is doubled, the arc is doubled.
3. If a central angle is doubled, the chord is doubled.
4. If an arc is doubled, the chord is not doubled.
5. Only two circles can have two points in common.

C. Problems and Exercises

Prove:

1. If, from any point D in the circumference of a circle O, chords DA and DB are drawn making angle ODA = angle ODB, then chord DA = chord DB. (Omit reasons.)

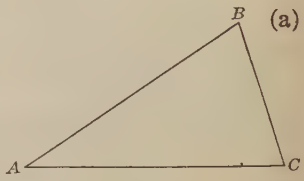


2. The line joining the point of intersection of two equal chords to the center of the circle bisects the angle formed by the chords.

3. Given circle O with chord AB = chord CD. (Use figure at right.)

To prove $AC = BD$. (Omit reasons.)

4. A line drawn through the intersection of the diagonals of a parallelogram is terminated by the two opposite sides. Prove that this line is bisected at the point of intersection of the diagonals. (Omit reasons.)



5. Fill in the blank spaces with correct statements. Use figures as indicated by letters.

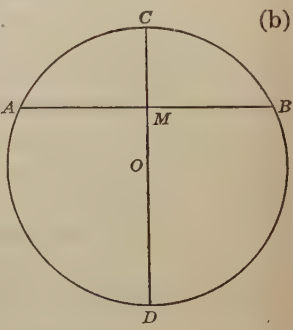
(a)

Given $AB = AC$. We can prove that ——.

(b)

Given $AB \perp$ diameter CD, $AM = MB$.

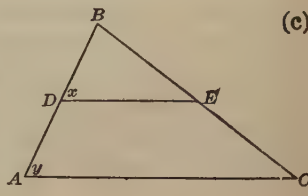
We can prove that ——.



(c)

Given $BD = DA$, $\angle x = \angle y$.

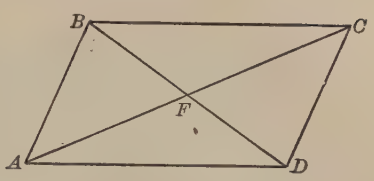
We can prove that ——.



6. State whether the following is true or false.

Given parallelogram $ABCD$.

We can prove $AF = FD$.



TEST VII

(To follow the completion of Proposition XXI, Book II)

A. Statements to Be Completed

1. Given AB and ES tangents to the circle; O is the center; $BROE$ is a straight line; AR, AD, RD, and ED are chords.

Then:

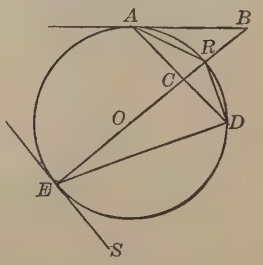
(a) $\angle RDE$ contains —— degrees.

(b) If arc $AR = 50°$, $\angle B =$ ——.

(c) If arc $AR = 50°$, and arc $RD = 40°$, $\angle BCD =$ ——.

(d) If arc $AR = 50°$, and arc $RD = 40°$, $\angle BAD =$ ——.

(e) $\angle BES$ contains —— degrees.



2. Given parallel chords AE and BC in circle O; BC extended to P; PA intersects the circle at D; BOE and AOC are straight lines.

Then:

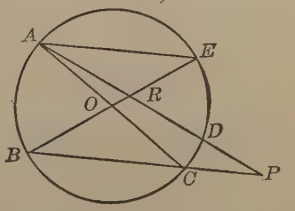
(a) If $\angle CAD = 10°$, arc CD has ——.

(b) If $\angle CAE = 35°$, arc CE has ——.

(c) If $\angle CAD = 10°$, and arc BA has $70°$, $\angle P =$ ——.

(d) If arc BA has $70°$, $\angle BOC =$ ——.

(e) If arc BA has $70°$ and $\angle CAD = 10°$, $\angle ARE =$ ——.

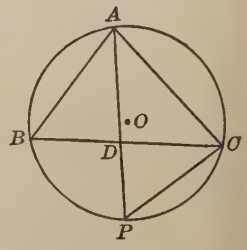


B. True-False Statements

1. Equal central angles have equal arcs.
2. An angle inscribed in a semicircle is a right angle.
3. A parallelogram inscribed in a circle is a square.
4. If a chord is doubled, the central angle is also doubled.
5. If an angle inscribed in a circle is 40°, its intercepted arc is 20°.
6. Angles inscribed in the same segment are equal.
7. A radius perpendicular to a chord of a circle bisects the chord.
8. The center of a circle inscribed in a triangle is equidistant from the three sides.
9. From any point on a circle only one diameter can be drawn.
10. In a circle, the chord of an arc of 60° is twice as long as the chord of an arc of 30°.

C. Problems and Exercises

1. Given $\triangle ABC$ inscribed in the circle with center O; AP bisects arc BC and intersects chord BC at D; chord PC is drawn.

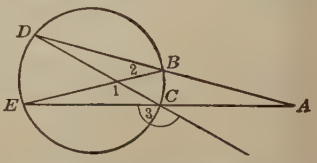


To prove the corresponding angles of $\triangle ABD$ and APC are equal.

2. Complete and prove the following: An angle formed by two chords intersecting within a circle is measured by ——. (Omit reasons.)

3. Complete and prove the following: An inscribed angle one of whose sides is a diameter is measured by ——. (Omit reasons.)

4. Given circle $DBCE$ in which $\overset{\frown}{DE} = 60°$, $\overset{\frown}{BC} = 30°$.



How many degrees are there in $\angle A$, $\angle 1$, $\angle 2$, $\angle 3$?

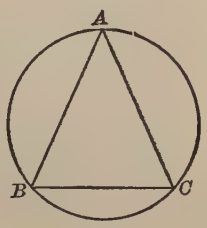
5. Two circles have a common center. A diameter AB is drawn in the larger circle, and a diameter PQ is drawn in the smaller one. Prove AQ is parallel to BP.

TEST VIII

(To follow the completion of Book II)

A. Statements to Be Completed

1. If one side of an inscribed triangle is a diameter of a circle, the triangle is a —— ——.



2. In this figure, $\overset{\frown}{AB} = \overset{\frown}{AC}$, $\overset{\frown}{BC} = 100°$. The number of degrees in $\angle B$ is ——.

3. A line perpendicular to a radius at its outer extremity is a ——.

4. —— circles can be drawn through two points.

5. Finding the point in a triangle equidistant from the three sides involves bisecting the ——.

6. The bisectors of the angles of a triangle meet in a —— which is equidistant from the ——.

B. True-False Statements

1. An angle is a point where two lines meet.

2. If two circles intersect, their line of centers is greater than the sum of their radii.

3. A line bisecting a chord of a circle will also bisect the arc which the chord determines.

4. Equal chords determine equal central angles in a circle.

5. A parallelogram is a quadrilateral whose sides are equal.

C. Problems and Exercises

1. (a) Prove that the tangents drawn to a circle from an outside point are equal.

(b) State two other conclusions that you can make from the diagram drawn for (a).

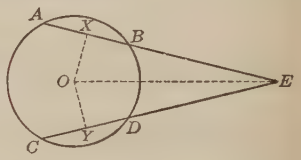
2. (a) From a given external point, construct a tangent to a given circle.

(b) Construct a tangent to a circle when the given point is on the circumference.

3. Prove that the bisectors of the opposite angles of a parallelogram are parallel.

4. $ABCD$ is a trapezoid inscribed in a circle, AB and DC being the bases. Arc AB measures 160° and arc DC measures 100°. Find one angle of the trapezoid and one angle formed at the intersection of the diagonals AC and BD.

5. If ABE and CDE are secants of a circle which make equal angles with the line from E to the center of the circle, the chord AB is equal to chord DC.



TEST IX

(To follow the completion of Proposition IV, Book III)

A. Review Questions

1. The first ratio of a proportion is $\frac{2}{3}$. What are some fractions which could form the second ratio?

2. What can be proved about a line which divides two sides of a triangle proportionally?

3. What ratio have corresponding sides of congruent triangles?

4. A mean proportional is usually written as which two terms of a proportion?

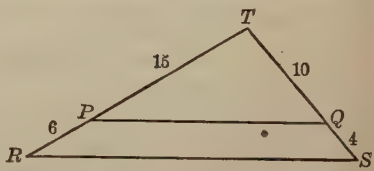
5. How many terms has a proportion?

B. Problems and Exercises

1. Construct a fourth proportional to the lines a, b, and c

2. Given the line RS. Divide RS into segments proportional to a, c, $2b$.

3. The difference between two lines is 8, and their ratio is $\frac{2}{5}$. Find each line.



4. Given PQ parallel to RS.

State what proportions may be written beginning with

(a) 6, (b) 4, (c) 10, (d) 21, (e) 14.

5. Prove that a line drawn through the midpoint of one side of a triangle parallel to the median to a second side cuts off $\frac{1}{4}$ of the side to which the median is drawn.

TEST X

(To follow the completion of Book III)

A. Statements to Be Completed

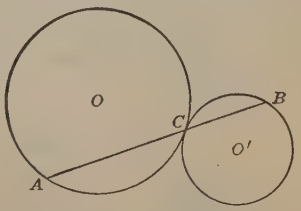
1. The bisectors of the opposite angles of a parallelogram are ——.

2. In the triangle ABC, the line DE is drawn parallel to AB. If $AB = 10''$, $AC = 8''$, $BC = 6''$, and $DE = 5''$, then $CD =$ —— and $CE =$ ——.

3. Similar polygons are polygons whose corresponding sides are —— and whose corresponding angles are ——.

4. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments with ratio 9 to 4. The altitude is ——.

5. If two circles are tangent externally at C, and ACB is drawn through C as in the annexed figure, then $\frac{AC}{CB} =$ ——.



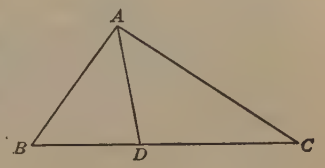
6. If a and $2b$ are given lines, then $\sqrt{2ab}$ is called ——.

7. If two triangles have their angles respectively equal, they are ——.

8. Two rectangles are similar if —— —— are ——.

B. True-False Statements

1. If in the $\triangle ABC$ having $AC > AB$, AD, the bisector of angle A, is drawn, meeting BC in D, then $DC > DB$.



2. The base of a triangle is 10 ft. The other two sides are 8 ft. and 5 ft. A line parallel to the base cuts 1 ft. from the lower end of the shorter side. The length of the line parallel to the base is 2 ft.

3. Two isosceles triangles in which the vertex angles are equal are similar.

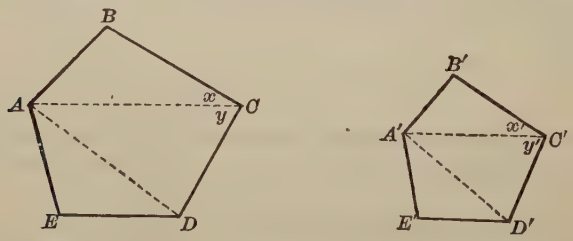
4. If two polygons of the same number of sides have their corresponding sides proportional, they are similar.

5. The diagonals of a trapezoid divide each other proportionally.

C. Problems and Exercises

Directions: Answer only the questions asked. Do not give complete proofs except where asked. If the diagram is given, it need not be copied. This applies also to what is given and what you are to prove.

1. Make a diagram, mark all the angles, write the three sets of equal ratios, and then circle the set that is discarded in proving: In a right triangle the perpendicular on the hypotenuse is the mean proportional between the segments of the hypotenuse.



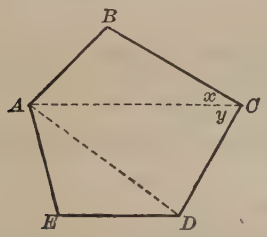
2. Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.
Assume $\triangle ABC \sim \triangle A'B'C'$.
To prove, giving reasons, $\triangle ACD \sim \triangle A'C'D'$.

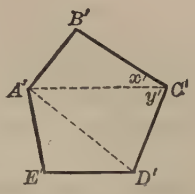
3. Given tangent PC, and secant PA intersecting a circle in F. Prove $AP : PC = PC : PF$. Make diagram, state which triangles are to be proved similar, mark the angles, and write the sets of equal ratios.

4. Given triangles ABC and $A'B'C'$, angle A = angle A', angle B = angle B', angle C = angle C'. Prove triangle ABC similar to triangle $A'B'C'$. (Omit reasons.)

5. Given triangle ABC with angle B bisected by BD.

Prove $\frac{AD}{DC} = \frac{AB}{BC}$. (Omit reasons.)





6. Given $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, $\triangle ADE \sim \triangle A'D'E'$.

To prove, polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

State two reasons for the equality of the following ratios:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{DE}{D'E'} = \frac{AE}{A'E'}.$$

7. Use the diagram for Ex. 6:

Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

To prove perimeter of $ABCDE$: perimeter $A'B'C'D'E' = AB : A'B'$.

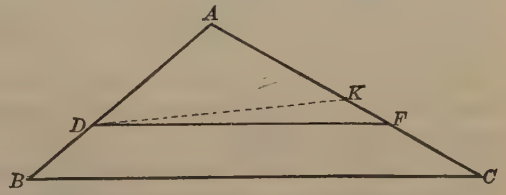
Give the reason for the step:

$$\frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AB}{A'B'}.$$

8.

Given $\frac{AB}{AD} = \frac{AC}{AF}$.

To prove $DF \parallel BC$.



Draw $DK \parallel BC$ through D.

State the reason for the step $\therefore AF = AK$.

TEST XI

(To follow the completion of Proposition IV, Book IV)

A. Statements to Be Completed

1. A parallelogram whose base is 12 is equal in area to a square whose side is 6. The altitude of the parallelogram is ——.

2. If a parallelogram and a square are equal in area, the side of the square is a —— between the base and altitude of the parallelogram.

3. If the base remains the same, doubling the altitude of a parallelogram —— the area.

B. True-False Statements

1. As the angle between two sides of a parallelogram increases, the area of the parallelogram increases.

2. If the side opposite the 30° angle of a right triangle is equal to 8 in., then the side opposite the 60° angle is equal to 16 in.

3. If two triangles are similar, their perimeters are to each other as any two corresponding sides.

C. Problems and Exercises

Directions: For Questions 1–4: (a) make diagram, (b) write "Given" and "To prove," (c) state which triangles must be proved similar, (d) mark the angles, (e) write the proportions.

1. QT and PS are altitudes in a triangle PQR. Prove $PS \cdot QR = QT \cdot PR$.

2. In a triangle ABC, P, Q, and R are the midpoints of sides AB, BC, and AC respectively. Prove triangle PQR is similar to triangle ABC.

3. In a circle two chords AB and CD intersect at E, C being the midpoint of the minor arc AB. Draw AC and prove that $\overline{AC}^2 = CE \cdot CD$.

4. A triangle ABC is inscribed in a circle. Chord MN is drawn parallel to the tangent at B, meeting AB in P and BC in Q. Prove $AB \cdot BP = BQ \cdot BC$.

5. Given the segments a and b. Construct $x = \sqrt{2ab}$.

6. The hypotenuse of a right triangle is 15 in., and one leg is 12 in. Find: (a) the segments made by the altitude upon the hypotenuse, (b) the altitude upon the hypotenuse, (c) the third side.

Test XII

(To follow the completion of Book IV)

A. Statements to Be Completed

1. If one triangle has half the base and twice the altitude of another triangle, their areas are ——.

2. A square constructed on the diagonal of a given square is —— the given square in area.

3. If the base of a parallelogram is increased and the altitude is unchanged, the area is ——.

4. If the base and altitude of a triangle remain the same, but the other two sides are changed, the area —— ——.

5. The line joining the midpoints of two sides of a triangle divides it into two parts whose areas are to each other as —— is to ——.

6. If the diagonal of a square equals $3\sqrt{2}$, then the area of the square is ——.

7. The difference between the square of the —— of a right triangle and the square of one of the —— is equal to the —— of the other ——.

8. The areas of the squares constructed on the altitude and the base of an equilateral triangle are to each other as —— is to ——.

B. True-False Statements

1. The diagonals of a parallelogram divide the parallelogram into four equivalent triangles.

2. A triangle is equivalent to twice a rectangle having the same base and altitude.

3. The area of a trapezoid is equal to one-half the product of its altitude and its median.

4. Area is the number of square inches a thing has.

5. The product of the legs of a right triangle equals the product of the hypotenuse and the altitude on the hypotenuse.

6. The area of an equilateral triangle with side x is $\frac{x^2\sqrt{3}}{4}$.

7. The area of a right triangle is equal to one half the product of the two legs.

8. Similar triangles are to each other as any two corresponding sides.

9. A triangle is equal to one half a parallelogram having the same base and altitude.

10. The area of a rhombus equals the sum of its diagonals.

11. If two triangles are similar, their areas are to each other as any corresponding sides.

C. Problems and Exercises

1. Prove that the area of a parallelogram is equal to the product of its base and altitude.

2. Give proof. (Omit reasons.)

(a) Two triangles are similar if their corresponding angles are equal.

(b) From a given external point, the tangent to a circle is a mean proportional between the whole secant and its external segment.

(c) In a right triangle the hypotenuse squared equals the sum of the squares of the other two sides.

3. If, in a circle O, AB is a diameter, AD a tangent with F any point in AD, secant BD cuts the circle in C, and secant BF cuts the circle in E, prove $BE{:}BD = BC{:}BF$.

4. Construct a square equal to the sum of two given squares.

5. The base and altitude of a triangle are 12 and 20 in., respectively. 6 in. from the base a line parallel to the base is drawn. Find the area of the trapezoid thus formed.

6. Median BD in $\triangle ABC$ is drawn. If E is any point in BD, prove that $\triangle ABE$ equals $\triangle BEC$ in area.

Test XIII

(To follow the completion of Book V)

A. Statements to Be Completed

1. If a is the apothem and s the side of a regular hexagon, its area is ——.

2. Points A, B, C, D, E, and F divide a circle into equal arcs. A chord joining two successive points equals the ——.

3. A tangent at one of the vertices of a regular inscribed pentagon makes an angle of —— degrees with one of the sides.

4. The midpoints of the sides of a regular polygon lie on a circle whose radius is the —— of the polygon.

5. If 8 in. is the radius of a circle, the side of a square inscribed in the circle is equal to ——, and the side of an equilateral triangle inscribed in the circle is equal to ——.

6. The area of an inscribed equilateral triangle is equal to —— the area of a regular hexagon inscribed in the same circle.

7. The area of a square inscribed in a circle is to the area of the circle as —— is to ——.

8. The ratio of the area of a circle to its circumference is ——.

9. The hypotenuse of a right triangle inscribed in a circle is the —— of the circle.

10. A regular polygon whose angle at the center is 18° has —— sides.

B. True-False Statements

1. The radius drawn to any vertex of a regular polygon bisects that angle of the polygon.

2. The central angle of a regular polygon is always less than 120°.

3. If the number of sides of a regular inscribed polygon is doubled, the area is increased.

4. Areas of similar sectors are to each other as their radii.

5. If the central angles of a regular polygon are bisected, and if the points of intersection of bisectors and sides are joined, there will be formed a regular polygon of double the number of sides of the original polygon.

6. The apothem of a regular polygon is the radius of the circumscribed circle.

7. The ratio of the circumference of a circle to its radius is $2\pi : 1$.

8. The radii of a regular polygon divide the polygon into congruent triangles.

C. Problems and Exercises

For a circle of radius 12 in. find:

1. The area of a regular inscribed triangle.
2. The area of a regular circumscribed hexagon.
3. The length of an arc of 40°.
4. The side of a square whose area equals that of the circle.
5. The area of the inscribed square.

Test XIV—Survey Test

Part I

Directions: Complete the following statements.

1. If sides AB and CD of quadrilaterial $ABCD$ are equal and parallel, then $ABCD$ must be a ——.

2. When two lines are cut by a transversal so that a pair of —— angles are equal, the two lines are parallel.

3. The sides of a right triangle are 3, 4, and 5. Its area is ——.

4. The hypotenuse of a right triangle whose legs are 5 and 12 is ——.

5. If a is the apothem and s the side of a regular hexagon, its area is ——.

6. If in similar triangles ABC and $A'B'C'$, $AB : A'B' = 3 : 2$, then triangle ABC : triangle $A'B'C' =$ —— : ——.

7. If the parallel sides of a trapezoid are 12 and 16, and the altitude is 4.5, the area of the trapezoid is ——.

8. If D and E are points in the sides AC and BC of triangle ABC such that $CD : DA = CE : EB$, then triangle CDE is ——— to triangle ABC.

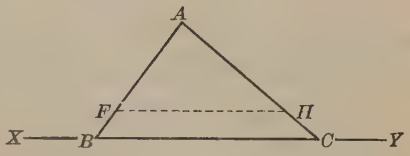
9. CD is the altitude on the hypotenuse of the right triangle ABC. If $AD = 30$ and $BD = 3\frac{1}{3}$, then $CD =$ ———.

10. Chords AB and CD of a circle intersect at P. If $AP = 6$, $CP = 8$, $BP = 16\frac{2}{3}$, then $DP =$ ———.

11. ABC is a triangle inscribed in a circle. If angle $A = 40°$, angle $B = 65°$, then the arc AB contains ——— degrees.

12. Points A, B, C, D, and E divide a circle into equal arcs. Chords joining successive points of division form a ———.

13. In this figure, if angle HFB = angle FBX, then line FH ———.



14. In the figure, if FH is parallel to XY, then angle YCH = angle ———.

15. In the figure of Question 14, if angle AFH = angle FBC, then triangle AFH is ——.

16. A circle can be circumscribed about a —— polygon.

17. In triangle ABC, if AB is greater than BC is greater than AC, the smallest angle is angle ———.

18. In triangle ABC, if angle A is less than angle B is less than angle C, the longest side is side ———.

19. In triangle ABC, angle A = twice angle B, and angle B = twice angle C; then angle A contains ——— degrees.

20. If two isosceles triangles have a common base, the line determined by their vertices is ———.

21. The center of the circle inscribed in a triangle is equidistant from the three ———.

22. The diameter DC of a circle is perpendicular to chord AB. If $\overset{\frown}{AC}$ contains 70°, then $\overset{\frown}{BD}$ contains ——— degrees.

23. The distances of two equal chords from the center of a circle are ———.

24. A line is parallel to one side of a triangle if it divides the other two sides ———.

25. Two triangles in which the ratios of corresponding sides are equal are ———.

26. The area of a circle whose radius is $2a$ is ———.

27. The center of the circle circumscribed about a given triangle is the point of intersection of the ———.

28. If the vertex angle of an isosceles triangle contains $2a$ degrees, each of the other two angles contains ——— degrees.

29. Pi is the ratio of the ——— to the ——— of the circle.

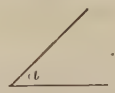
30. Each side of a square inscribed in a circle is equal to ——— times the radius of the circle.

31. The locus of points equidistant from the sides of a given angle is ———.

32. Opposite angles of an inscribed quadrilateral are ———.

In Questions 33–37 *leave all construction lines on paper.*

33. Bisect angle a shown here.



34. Construct the perpendicular bisector of segment AB.

A ——————— B

35. Construct the perpendicular to AB from point P.

A ——————— P ——————— B

36. Construct the mean proportional between a and b shown here. a | b

37. Divide the line-segment below into *three* equal parts.

In the following, select the word in the parentheses that gives the correct answer:

38. "Through or connecting any two points, a straight line may be drawn" is a (definition, postulate, theorem).

39. "If two angles of a triangle are equal, the sides opposite these angles are equal" is a (definition, postulate, theorem).

40. "The point of intersection of the bisectors of the angles of a triangle is the *incenter* of the triangle" is a (definition, postulate, theorem).

Part II

Group 1

Answer *three* questions from this group.

1. Prove: If a perpendicular is erected at a point in the base of an isosceles triangle meeting one leg and the other leg produced, the triangle formed external to the given triangle is isosceles.

2. The bisector of angle C of the inscribed triangle ABC cuts AB in D and the circle in E. Prove $AC : CE = CD : BC$.

3. $ABCD$ is a parallelogram. On diagonal BD points E and F are taken so $BE = DF$. Prove $AECF$ is a parallelogram.

4. Prove that, if two diagonals of an inscribed regular pentagon intersect, the longer segment of either diagonal is equal to a side of the pentagon.

5. Find the locus of the centers of circles that pass through two given points.

Group 2

Answer *three* questions from this group.

6. The base of a triangle is 12 in. Two lines are drawn parallel to the base terminating in the other two sides and dividing the triangle into three equal parts. Find the lengths of these lines.

7. The sum of the diagonals of a rhombus is 23 in. and one diagonal is 7 in. longer than the other. Find the area and the perimeter of the rhombus.

8. A chord 1 ft. long is 4 in. from the center of a circle. How far from the center of this circle is a chord that is 9 in. long?

9. From a point P, $6\frac{1}{2}$ in. from the center of a circle whose diameter is 5 in., a secant and a tangent are drawn. Find the length of the tangent and the product of the secant and its external segment.

10. ABC is a triangle inscribed in a circle whose radius is 2 in. AB is a diameter of the circle and angle ABC is 30°. Find the area of the triangle.

Test XV—Survey Test

Part I

Directions: Answer all questions in this part. Each correct answer will receive 2 credits; no partial credit allowed.

In Questions 1–20 fill the blanks so as to make an accurate statement in each case.

1. The fourth proportional to 17, 51, and 3 is ———.

2. If the radius of a circle is 6, the apothem of a regular inscribed hexagon is ———.

3. In triangle ABC, AC equals BC, and AD bisects angle A. If angle ADC equals 105°, then angle C contains ——— degrees.

4. A median of any triangle divides it into two triangles that are ———.

5. In a circle, if the chord of an arc of 60° is 10, the distance from the center to the chord is ———.

6. Two triangles are similar and the first is $2\frac{1}{4}$ times the second. If a side in the first is 12, the corresponding side in the second is ———.

7. The base angle of an isosceles trapezoid is 60°, the upper base is 10, and the legs each 8. Find its area.

8. Angle BAC is inscribed in a circle, the arcs AB, BC, and CA being in the ratios 3 : 2 : 4. Then angle A contains ——— degrees.

9. A tangent is drawn at a vertex of an inscribed regular pentagon. This tangent makes an angle of ——— degrees with one side.

10. The diameter DC of a circle is perpendicular to chord AB. If arc AC contains 50°, arc BD contains ——— degrees.

11. Obtuse triangle ABC has AC equal to BC. Then ——— is the longest side.

12. If two triangles have their three angles equal respectively, the triangles are always ———.

13. A chord perpendicular to a diameter divides the diameter into segments $2\frac{1}{2}$ in. and 10 in. long. The length of the chord is ———.

14. In parallelogram $ABCD$, E is the midpoint of side BC. Triangle DEC is exactly one ——— of parallelogram $ABCD$.

15. If the legs of a right triangle are 12 and 16, and the altitude on the hypotenuse is drawn, the segment adjacent to 12 is ———.

16. CD and AE are medians of triangle ABC intersecting at O. If DE is drawn, then triangle AOC is similar to triangle ———.

17. The locus of points 4 in. from a circle whose radius is 3 in. is a ———.

18. If two tangents to a circle from an external point make an angle of 40° with each other, the minor arc intercepted is an arc of ——— degrees.

19. The bisectors of the angles of a triangle meet in a point which is the center of the ——— circle.

20. If the circumference of a circle is 16π, the area is ———.

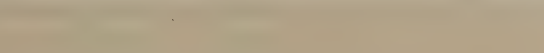
21. Name three of the following parts of parallelogram $ABCD$ that would always determine the parallelogram: AB, AC, BD, AD, angle A.

22. In the following statement select the word in the parentheses that gives the correct answer: "Two straight lines can intersect in but one point" is a (definition, postulate, theorem).

In Questions 23–25 *leave all construction lines on the paper.*

23. Inscribe a square in a given circle O.

24. Divide line AB into two parts in the ratio $a : b$.



25. Construct a tangent to a given circle O from a given external point P.

Part II

Group 1

Answer *three* questions from this group.

1. E is any point in the common base of two triangles ABC and ABD. From E a parallel to AC meets BC in F and a parallel to AD meets BD in G. Prove that line FG is parallel to line CD.

2. A triangle ABC is inscribed in a circle, and a tangent is drawn at A. A line parallel to the tangent cuts AB in E and AC in D. Prove that triangle ADE is similar to triangle ABC.

3. If any point in one side of a triangle is joined to the midpoints of the other two sides, prove that the area of the quadrilateral thus formed is one half of the triangle.

4. If from a fixed point in a triangle lines are drawn to all points in the sides of the triangle, what is the locus of the midpoints of these lines? Prove your answer.

5. $MNOP$ is a parallelogram, and parallel lines are drawn from P and N meeting diagonal MO at R and S, respectively. Prove that PR equals NS.

Group 2

Answer *two* questions from this group.

Note: Irrational results may be left in the form of π and radicals.

6. The legs of a right triangle are in the ratio 3 : 4, and its area is 294. What is the length of the hypotenuse?

7. Find the area of the following plot of ground: Starting at A, go in a straight line 10 miles to B; turn 60° to left and go in a straight line 6 miles to C; turn 90° to left and go in a straight line 8 miles to D; then go in a straight line to A. (320 rd. = 1 mi.; 160 sq. rd. = 1 acre.)

8. $ABCD$ is a rectangular sheet of paper 6 in. $\times$ 12 in. If it is folded over so that point A falls on C, and the paper is creased, find the length of the crease.

Test XVI—Selection and Organization of Geometric Facts

Directions: Select and arrange from the facts given only those which form a proof for each problem. Follow given diagram and hypothesis.

Use given figure for Parts A, B, and C.

A. Given $\triangle ABC$ and $CD \perp AB$.

To prove $\angle A - \angle B = \angle BCD - \angle ACD$.

1. $\angle A + \angle ACD = 90°$.
2. $\angle A + \angle ACD = \angle B + \angle BCD$.
3. $\angle ADC = \angle BDC$.
4. $\angle A - \angle B = \angle BCD - \angle ACD$.
5. $\angle B + \angle BCD = 90°$.

B. Given $\triangle ABC$ and $\angle B = 30°$. (See Part A.)

To prove area of $\triangle ABC = \frac{1}{4}AB \times BC$.

Draw $CD \perp AB$. Other facts that are known:

1. $CD = \frac{1}{2}BC$.
2. $\angle DCB = 60°$.
3. Area of $\triangle ABC = \frac{1}{2}CD \cdot AB$.
4. $BD = CD\sqrt{3}$.
5. Area of $\triangle ABC = \frac{1}{4}BC \cdot AB$.
6. $CA > CD$.

C. Given $\triangle ABC$ and $CD \perp AB$. (See Part A.)

To prove exterior angle at $B = 90° + \angle DCB$.

Extend AB beyond B.

Facts known are:

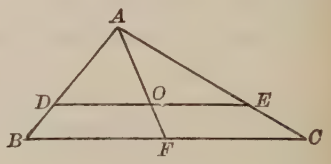
1. $\angle ACB = \angle ACD + \angle DCB$.
2. $\angle A + \angle B + \angle ACB = 180°$.
3. Exterior angle at $B = \angle A + \angle ACB$.
4. $\angle ACB > \angle B$.
5. Exterior angle at $B = 90° + \angle DCB$.
6. $\angle A + \angle B = 180° - \angle ACB$.
7. $\angle A = 90° - \angle ACD$.

Test XVII—Matching

Directions: In this test, copy the left-hand column from (a) through (f). Then place beside each statement the number of the reason (as listed in the right-hand column) which makes the statement true.

Given $\triangle ABC$ with the median AF and $DE \parallel BC$, intersecting AF at O.

To prove $OD = OE$.



(a) $\triangle ABF \sim \triangle ADO$	1. In similar triangles, corresponding sides are proportional.
(b) $\frac{AF}{AO} = \frac{BF}{DO}$	2. In two proportions, if three terms of one are equal to three corresponding terms of the other, the fourth terms are equal.
(c) $\triangle AFC \sim \triangle AOE$	3. By hypothesis.
(d) $\frac{AF}{AO} = \frac{FC}{OE}$	4. A line parallel to base of a triangle cuts off a triangle similar to given triangle.
(e) $BF = FC$	
(f) $OD = OE$	

Test XVIII—Converse of a Statement

State the converses of the following theorems:

Note: Not all converses are true.

1. If the bisector of the exterior angle at C of a triangle ABC is parallel to AB, then AC is equal to BC.

2. The median to the hypotenuse of a right triangle is equal to one half the hypotenuse.

3. If the radius of the circumscribed circle and a side of a triangle are given, the angle opposite the side is determined.

4. If two triangles are similar, their corresponding sides are proportional.

5. A triangle can be inscribed in a circle.

6. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other sides.

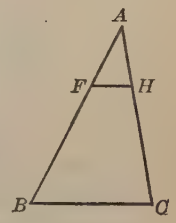
7. The diagonals of a square are equal.

8. In a 30°, 60° right triangle, the side opposite the 30° angle is one half of the hypotenuse.

Test XIX—Drawing Conclusions from Diagrams

Complete the following statements:

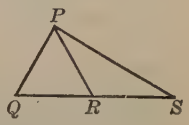
1. Given $\triangle ABC$ with $\frac{AF}{AB} = \frac{AH}{AC}$.



We can prove (*a*) that ———, and (*b*) that ———.

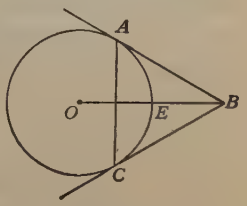
2. If $AF = \frac{1}{2}FB$, and $AH = \frac{1}{2}HC$ (using the figure of Question 1), we can prove that $\triangle AFH$ ———.

3. Given PR a median in $\triangle PQS$.



We can prove that ———.

4. Given a circle O, AB and BC tangents, AC the chord of contact, and OB the line joining the center to B.

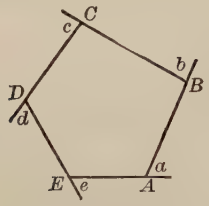


We can prove that (*a*) ———, (*b*) ———, (*c*) ———, (*d*) ———.

APPENDIX

Proposition I. Theorem

416. *The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is equal to four right angles.*



Given polygon $ABCD$ . . . with its sides produced in succession, forming the exterior $\measuredangle a$, b, c, etc.

To prove $\angle a + \angle b + \angle c$ . . . $=4$ rt. $\measuredangle$.

Proof. 1. Through any convenient point O, draw lines $\|\|AB$, BC, CD, . . . respectively, and extending in the same directions as these sides, and forming $\measuredangle a'$, b', c', . . .	1. § 100.
2. Then $\angle a = \angle a'$, $\angle b = \angle b'$, $\angle c = \angle c'$, etc.	2. Two $\measuredangle$ are = when their sides are $\|\|$ right to right and left to left. (§§ 118, 119.)
3. $\therefore \angle a + \angle b + \angle c + \ldots = \angle a' + \angle b' + \angle c' + \ldots$	3. Why?
4. But $\angle a' + \angle b' + \angle c' + \ldots = 4$ rt. $\measuredangle$.	4. Why?
5. $\therefore \angle a + \angle b + \angle c + \ldots = 4$ rt. $\measuredangle$.	5. Ax. 1. Q.E.D.

Proposition II. Problem

417. *On a given straight line as a base, to construct a circular segment in which a given angle can be inscribed.*

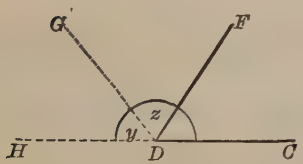


Fig. 1

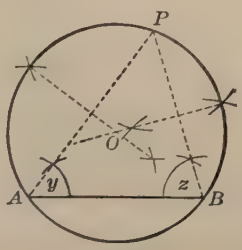


Fig. 2

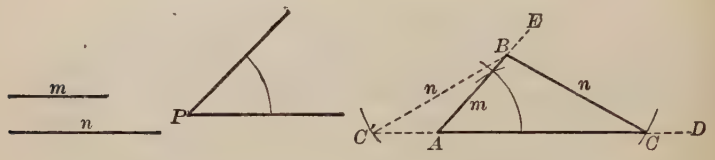
Given the straight line AB and $\angle CDF$.

To construct on AB a segment of a circle in which $\angle CDF$ can be inscribed.

Construction. 1. Produce CD to H, and from D draw DG, dividing $\angle HDF$ into two convenient angles, y and z.	1. Post. 1.
2. From A draw line AP making $\angle BAP = \angle y$.	2. § 90.
3. From B draw line PB making $\angle PBA = \angle z$.	3. § 90.
4. Then PA and PB intersect at some point P.	4. § 104.
5. About $\triangle APB$ circumscribe a circle.	5. § 257.
Then APB is the segment required.	
Proof. 1. $\angle P$ is sup. of $\angle y + \angle z$ (Fig. 2).	1. § 108.
2. But $\angle CDF$ is sup. of $\angle y + \angle z$ (Fig. 1).	2. § 32.
3. $\therefore \angle P = \angle CDF$.	3. § 66. **Q.E.F.**

Proposition III. Problem

418. *To construct a triangle, given two sides and an angle opposite one of them.*



Given m and n two sides of a $\triangle$, and $\angle P$ opposite n.

To construct the triangle.

Construction. Several cases occur, according to the relative size of the given sides and the size of the given angle.

Case I. *When $n > m$ (and $\angle P$ is acute).*

1. At the point A construct $\angle DAE = \angle P$.	1. § 90.
2. On AE mark off AB equal to m.	2. Post. 2.
3. With B as a center and with a radius equal to n, describe an arc intersecting AD at C and C′.	3. Post. 3.
4. Draw BC and BC'.	4. Post. 1.
5. Two $\triangle$, ABC and ABC', are obtained, containing the sides m and n; but only one of the $\triangle$, $\triangle ABC$, contains $\angle P$.	5. Hyp.
Hence ABC is the $\triangle$ required.	

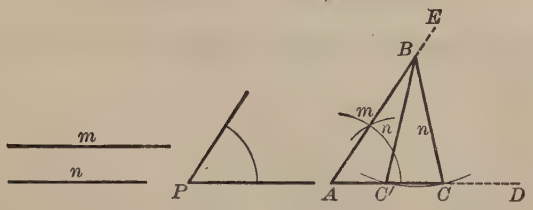
Case II. *When $n = m$ (and $\angle P$ is acute).*

Make the same construction as in Case I.

The arc drawn intersects the line AD in the points A and C.

Hence an isosceles $\triangle ABC$ is obtained, which is the triangle required.

CASE III. *When $n<m$ (and $\angle P$ is acute).*



Make the construction in the same way as in Case I.

Two $\triangle$, ABC and ABC', are obtained, each of which contains the sides m and n and an angle equal to $\angle P$ opposite the side n.

$\therefore$ $\triangle ABC$ and ABC' are the triangles required. Q.E.F.

Discussion. In Case I, if $\angle P$ is a right $\angle$, let the student construct the figure and show that there are two $\triangle$ answering the given conditions. If $\angle P$ is an obtuse angle, let him construct the figure and show that there is but one answer.

In Case II, if $\angle P$ is right, or obtuse, what results are obtained?

In Case III, if $\angle P$ is acute and n equals the $\perp$ from B to AD, how many constructions are possible? Also, if n is less than this $\perp$, how many?

If $\angle P$ is right, or obtuse, what result is obtained?

Ex. 1. Construct a triangle in which two of the sides are 1 in. and $1\frac{1}{2}$ in., and the angle opposite the latter side is 45°.

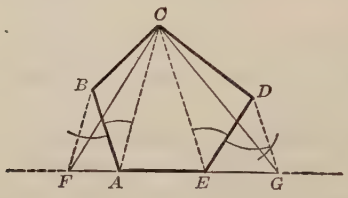
Ex. 2. Construct a triangle in which two of the sides are $1\frac{1}{2}$ in. and $1\frac{1}{4}$ in., and the angle opposite the latter side is 45°.

Ex. 3. Construct a triangle in which two of the sides are $1\frac{1}{2}$ in. and $\frac{3}{4}$ in., and the angle opposite the latter side is 30°.

Ex. 4. Make up the data and construct a triangle in which a given angle is acute and there are two solutions. No solution.

Proposition IV. Problem

419. *To construct a triangle equal in area to a given polygon.*



Given the polygon $ABCDE$.

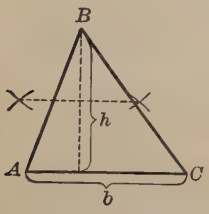
To construct a triangle equal in area to $ABCDE$.

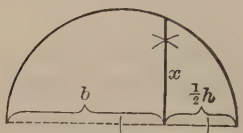
Construction. 1. Take any three consecutive vertices, A, B, C, and draw AC.	1. Post. 1.
2. Draw $BF \parallel AC$ and meeting EA produced at F. Draw FC.	2. § 100, Post. 1.
3. In the polygon $FCDE$ take the three consecutive vertices, C, D, E, and draw CE.	3. Post. 1.
4. Draw $DG \parallel CE$ and meeting AE produced at G. Draw CG.	4. Why?
Then $\triangle FCG$ is the triangle required.	
Proof. 1. Taking AC as their common base, $\triangle ABC = \triangle AFC$.	1. § 344.
2. $\triangle ACE = \triangle ACE$.	2. Ident.
3. Taking CE as their common base, $\triangle CDE = \triangle CGE$.	3. Why?
4. Adding, $\triangle ABC + \triangle ACE + \triangle CDE = \triangle AFC + \triangle ACE + \triangle CGE$,	4. Why?
or polygon $ABCDE = \triangle FCG$.	Q.E.F.

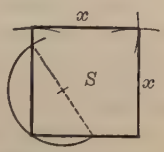
Ex. Construct a quadrilateral $ABCD$ in which $AB = 2$ in., $BC = 1\frac{1}{2}$ in., $CD = 2\frac{1}{4}$ in., $AD = 2\frac{1}{2}$ in., $AC = 2\frac{1}{2}$ in. Then construct a triangle equal in area to the quadrilateral.

Proposition V. Problem

420. *To construct a square equal in area to a given triangle.*







Given $\triangle ABC$ with base b and altitude h.

To construct a square equal in area to $\triangle ABC$.

Construction. 1. In the $\triangle ABC$, bisect the altitude h.	1. § 130.
2. Construct x, the mean proportional between b and $\frac{1}{2}h$.	2. § 320.
3. Construct S, a square whose side is x. Then S is the square required.	3. §§ 252, 80.
Proof. 1. Area $\triangle ABC = \frac{1}{2}bh$.	1. § 342.
2. $S = x^2 = \frac{1}{2}bh$.	2. §§ 277, 336.
3. $\therefore$ Square $S = \triangle ABC$.	3. Ax. 1.

Q.E.F.

421. Cor. *To construct a square equal in area to a given polygon.*

Use §§ 419, 420.

Ex. 1. Construct a triangle equal in area to a given hexagon.

Ex. 2. Construct a square equal in area to a given parallelogram.

Ex. 3. The sum of the areas of two similar triangles is 255 sq. in. Also the ratio of the areas of the triangles is 1 : 4. Find the area of each triangle.

Ex. 4. The length of a pavement is l feet and the width is w feet. If a brick is a inches long and b inches wide, obtain a formula for N, the number of bricks in the pavement.

METHOD OF LIMITS

422. A **variable quantity,** or a **variable,** is a quantity which may have an indefinite number of different values under the conditions of a theorem or problem.

Thus the distance a railroad train goes varies with the number of hours which the train travels.

423. A **constant** is a quantity which remains unchanged in value in a given problem or discussion.

Thus, if a polygon is inscribed in a circle and the number of sides of the polygon is doubled, quadrupled, etc., the perimeter of the polygon will be variable, but the circumference of the circle will remain unchanged; hence it is a constant.

424. The **limit** of a variable quantity is a constant quantity which, under the conditions of a given discussion, the given variable may approach as nearly as we please in value, but which the variable can never equal.

Thus, in the example given under § 423, the circumference of the circle is the limit of the perimeter of the inscribed polygon; also, the area of the circle is the limit of the area of the inscribed polygon.

The symbol for "approaches as a limit" is $\longrightarrow$.

425. Use of variables and limits.—Many of the properties of limits are the same as the properties of the variables approaching them. Hence a demonstration of a difficult theorem may often be obtained by first finding the properties of relatively simple variables and then transferring these properties to their more complex limits.

426. Properties of variables and limits.

1. *The limit of* a *times a variable equals* a *times the limit of the variable,* a *being a constant.* For, if the difference between a variable and its limit may be made as small as we please, a times this difference may be made as small as we please.

This principle expressed in symbols is as follows:

If v is a variable and c its limit and $v \longrightarrow c$, then $av \longrightarrow ac$.

2. *The limit of $\frac{1}{a}$th part of a variable is $\frac{1}{a}$th part of the limit of the variable,* **a** *being a constant.* For, if the difference between a variable and its limit may be made as small as we please, $\frac{1}{a}$th part of this difference may be made as small as we please.

That is, if $v \longrightarrow c$,

then, $\frac{v}{a} \longrightarrow \frac{c}{a}$.

3. *If two variables are always equal, and each variable approaches a limit, the limits are equal.* For, if the limits were unequal, the two variables would become unequal when they approached close to their respective limits.

Thus, if v and v' are two variables, and c and c' are their respective limits, the principle just stated, when expressed in symbols, is as follows:

If $v \longrightarrow c$, $v' \longrightarrow c'$, and $v = v'$,

then $c = c'$.

427. The circle and its parts as limits.—By methods which are beyond the scope of this book, it may be shown that

1. *If the number of sides of a regular polygon inscribed in a circle is increased indefinitely, the apothem of the polygon will approach the radius of the circle as its limit.*

2. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely.*

3. *The area of a circle is the limit which the areas of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely.*

By the method of limits, § 392 may be proved as follows:

Proposition VI. Theorem

(§ 392 proved by the method of limits.)

428. *The area of a circle is equal to half the product of its circumference by its radius.*

Given a ⊙ with its circumference denoted by C, radius by r, and area by A.

To prove $A = \frac{1}{2}rC$.

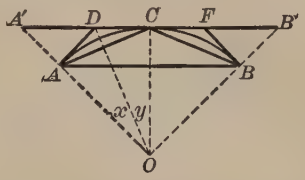
Proof. 1. Circumscribe a regular polygon about the given circle, and denote its perimeter by p, its area by A', and the number of its sides by n.	1. §§ 364, 375.
2. The apothem of the circumscribed regular polygon is the radius of the given circle, or r.	2. § 372.
3. $A' = \frac{1}{2}rp$.	3. § 388.
4. If n is increased indefinitely, $A' \to A$.	4. § 427, 3.
5. Also $p \to C$.	5. § 427, 2.
6. $\therefore \frac{1}{2}rp \to \frac{1}{2}rC$.	6. § 426, 1, 2.
7. In steps 4 and 6 we have the two variables, A' and $\frac{1}{2}rp$, and their respective limits, A and $\frac{1}{2}rC$. Of these, the two variables are equal.	7. Step 3.
8. Hence their limits are equal, or $A = \frac{1}{2}rC$.	8. § 426, 3. **Q.E.D.**

Ex. 1. The triangle ABC is inscribed in a circle. The chord AE is drawn perpendicular to BC, and the chord CD perpendicular to AB. Prove the arcs BD and BE equal.

Ex. 2. ACB and AHB are triangles on opposite sides of AB. R is a point on AB. RL is parallel to AC and meets BC in L. RK is parallel to AH and meets HB in K. Prove $LK \parallel CH$.

Proposition VII. Problem

429. *Given a circle, p the perimeter of a regular inscribed polygon and P the perimeter of a regular circumscribed polygon, each of the polygons having n sides, to find p' and P', the perimeters of the regular inscribed and circumscribed polygons, each of $2n$ sides, in terms of p and P.*



Given the circle O, AB a side of a regular inscribed polygon, and $A'B'$ a side of a regular circumscribed polygon (each of n sides); AC a side of a regular inscribed polygon, and DF a side of a regular circumscribed polygon (each of $2n$ sides), formed as in §§ 376, 377; the perimeters of the four polygons denoted by p, P, p', P', in order.

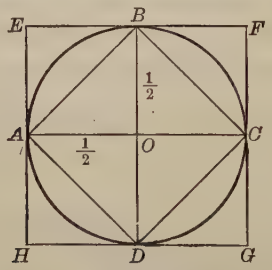
To find p' and P' in terms of p and P.

Solution. 1. $\frac{P}{p}=\frac{OA'}{OC}$.	1. §§ 378, 316.
2. $AD=DC$, $AO=CO$, $DO=DO$.	2. Why?
3. $\triangle ADO\cong\triangle DCO$.	3. Why?
4. $\angle x=\angle y$.	4. Why?
5. In $\triangle A'CO$, $\frac{OA'}{OC}=\frac{A'D}{DC}$.	5. § 297.
6. $\therefore$ from 1 and 5, $\frac{P}{p}=\frac{A'D}{DC}$.	6. Ax. 1.
7. $\therefore \frac{P+p}{p}=\frac{A'D+DC}{DC}=\frac{A'C}{DC}$ $=\frac{\frac{1}{2}A'B'}{\frac{1}{2}DF}=\frac{A'B'}{DF}=\frac{P}{n}\div\frac{P'}{2n}=\frac{2P}{P'}$.	7. § 285, Axs. 7, 9.

8. Hence $P+p : p=2P : P'$,	8. Ax. 1, § 277, Ax. 5.
and $P'=\frac{2Pp}{P+p}$. . . . . (1)	
9. $\angle DAC=\angle CAB$,	9. §§ 241, 247.
and $\angle DCA=\angle CBA$.	
10. $\therefore \triangle ACB \sim \triangle ADC$.	10. Why?
11. $AB : AC=AC : DC$.	11. Why?
12. Hence $\overline{AC}^2=AB\times\frac{1}{2}DF$, or $\left(\frac{p'}{2n}\right)^2=\frac{p}{n}\times\frac{P'}{4n}$.	12. Why?
13. Hence $p'^2=pP'$, and $p'=\sqrt{pP'}$. . (2)	13. Why? Q.E.F.

Proposition VIII. Problem

430. ***To compute approximately the numerical value of π.***



Given a circle O whose diameter is 1, and whose circumference is denoted by C.

To compute C approximately and hence find the approximate numerical value of π.

Solution. 1. In the circle O, inscribe the square $ABCD$, and circumscribe the square $EFGH$, and denote the perimeters of the two squares by p and P respectively.	1. §§ 362, 375.

2. $AB=\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2}=\frac{1}{2}\sqrt{2}$. 2. § 354, Ax. 6.

$\therefore p=2\sqrt{2}=2.82843^+$.

3. $EF=AC=1$. $\therefore P=4$. 3. § 155.

4. Denoting the perimeters of the regular inscribed and circumscribed octagons by p_8 and P_8 respectively, 4. § 429.

$$P_8=\frac{2Pp}{P+p}=3.31371^+,$$

and $p_8=\sqrt{pP_8}=3.06147^+$.

5. Similarly $P_{16}=\frac{2P_8p_8}{P_8+p_8}=3.18260^+$, 5. § 429.

and $p_{16}=\sqrt{p_8P_{16}}=3.12145^+$.

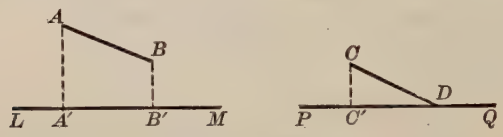
6. Proceeding in like manner, we obtain the following table: 6. § 429.

Number of Sides	Perimeter of Circumscribed Polygon	Perimeter of Inscribed Polygon
4	4	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

7. The result given in the last line in the table shows that the length of the circumference of the circle O, that is of C, lies between 3.14163^+ and 3.14157^+. Hence the approximate numerical value of π, when carried out to four decimal places, is 3.1416. 7. § 427, 2.

Q.E.F.

431. The **projection of a line** upon another given line is that part of the second line which is included between perpendiculars drawn from the extremities of the first line upon the second line. Thus the projection of AB on LM is $A'B'$; of CD on PQ, is $C'D$.



Ex. 1. A line 10 in. long makes an angle of 45° with a second line; find the projection of the first line on the second.

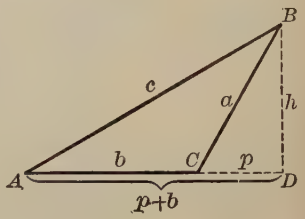
Ex. 2. Find the same, if the angle is 60°.

Ex. 3. If the side of an equilateral triangle is a, find its projection on the base.

EXERCISES: GROUP 91

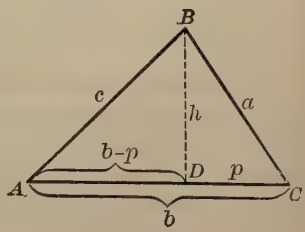
Ex. 1. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides by the projection of the other side upon it.

[Sug. In rt. $\triangle ABD$, $c^2=h^2+(p+b)^2$; also, in the rt. $\triangle CBD$, $h^2+p^2=a^2$, etc.]



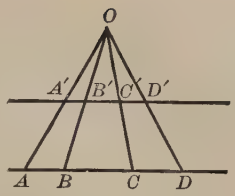
Ex. 2. In any oblique triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides by the projection of the other side upon it.

[Sug. In the rt. $\triangle ABD$, $c^2=h^2+(b-p)^2$; also in the rt. $\triangle BDC$, $h^2+p^2=a^2$, etc.]

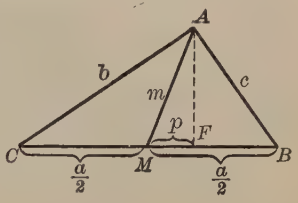


Ex. 3. If three or more lines pass through the same point and intersect two parallel lines, they intercept proportional segments on the parallel lines; that is, on the diagram,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}.$$



Ex. 4. If, in any triangle, a median is drawn to one side, the sum of the squares of the other two sides is equal to twice the square of half the given side, increased by twice the square of the median upon that side; that is, on the diagram, if AM is a median to BC, prove $b^2+c^2=\frac{a^2}{2}+2m^2$.

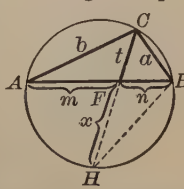


Show also that $m=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$.

[Sug. Apply Ex. 2 to $\triangle ABM$, and Ex. 1 to $\triangle AMC$.]

Ex. 5. In a parallelogram the sum of the squares of the sides equals the sum of the squares of the diagonals.

Ex. 6. The square of the bisector of an angle of a triangle is equal to the product of the sides forming the angle, diminished by the product of the segments of the third side formed by the bisector; that is, on the diagram show that $t^2=ab-mn$.

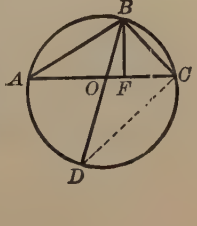


[Sug. Prove $\triangle AFC\sim\triangle HCB$, and hence that $b:t=x+t:a$. Use §§ 277, 321.]

Ex. 7. On the diagram of the preceding exercise, find the value of t in terms of a, b, c (the sides of the given triangle), and s, used as denoting $\frac{1}{2}(a+b+c)$.

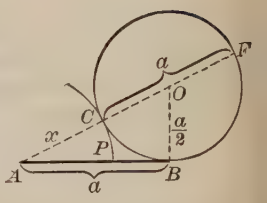
[Sug. This amounts to finding the value of m and n in terms of a, b, c. By § 297, $n:m=a:b$. Hence $m+n:m=b+a:b$ (§ 285), or $c:m=b+a:b$, etc.]

Ex. 8. In any triangle, the product of any two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side; that is, on the diagram prove $AB\times BC=DB\times BF$.



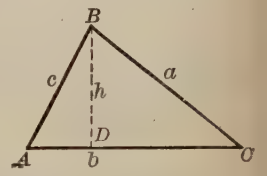
[Sug. Prove $\triangle ABF\sim\triangle DBC$, etc.]

Ex. 9. Divide a given straight line in extreme and mean ratio; that is, so that the whole line is to the larger part as the larger part is to the smaller part. Thus, on the diagram, divide AB so that $AB : AP = AP : PB$.



[Sug. At B erect the $\perp OB = \frac{1}{2}AB$ and draw the circle O with the radius OB. Use §§ 212, 323, etc.]

Ex. 10. If a, b, c, denote the sides of a triangle opposite the angles A, B, C, respectively, and $s = \frac{1}{2}(a+b+c)$, the area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.



[Sug. By Ex. 2, p. 350,

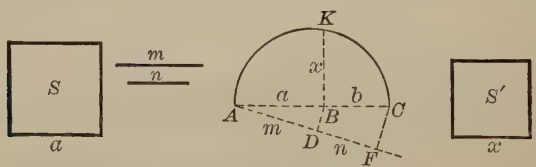
$$a^2 = b^2 + c^2 - 2b \times AD.$$

Hence $$AD = \frac{b^2+c^2-a^2}{2b}.$$

But, by § 355, $$h^2 = c^2 - \overline{AD}^2 = (c+AD)(c-AD).$$

$$\therefore h^2 = \left(\frac{2bc+b^2+c^2-a^2}{2b}\right)\left(\frac{2bc-b^2-c^2+a^2}{2b}\right), \text{ etc.}]$$

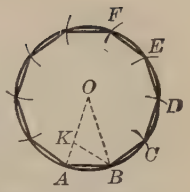
Ex. 11. Construct a square which shall have a given ratio to a given square.



[Sug. Given the square S, construct the diagram as indicated and obtain S', which is to S as $n : m$. Then, by use of §§ 318, 291, prove that $\frac{S}{S'} = \frac{a^2}{x^2} = \frac{a^2}{ab} = \frac{a}{b} = \frac{m}{n}$.]

Ex. 12. Construct a square which shall be to a given square as 3 : 2.

Ex. 13. Inscribe a regular decagon in a given circle.

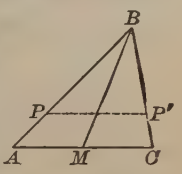


[Sug. Divide the radius OA at K so that $OA : OK = OK : KA$ (see Ex. 9, p. 352). Insert the chord $AB = OK$. Prove $\triangle AOB \sim \triangle AKB$ by § 308. Hence $AB = KB = KO$, and $\angle A = \angle AKB = 2\angle O = \angle OBA$. $\therefore \angle O = 36°$, etc.]

Ex. 14. By what per cent does 3.1416, the approximate value of π, differ from 3.141592, a closer approximation to the value of π?

Ex. 15. By what per cent does $\frac{22}{7}$ differ from 3.1416?

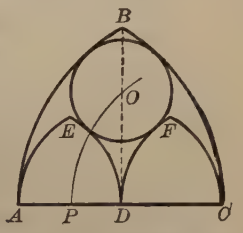
Ex. 16. It is evident that, if the median of a triangle (BM) is placed on a knife-edge, the triangle will balance (for if PP' is $\parallel AC$, the pull on P is balanced by the pull on P'). Hence find the center of mass for any triangle. For a regular pentagon.



It is useful to be able to determine the center of mass of an object by geometry, or by any other means, since a knowledge of the center of mass of a body often enables us to treat the body in a simple way, for example, as if the body were concentrated at a single point.

Ex. 17. If a box has a triangular end, subject to the same pressure at all points, at what single point on the end must a supporting pressure be applied?

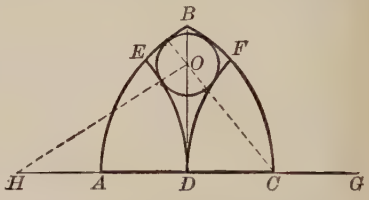
Ex. 18. D is the midpoint of the straight line AC. ABC, AED, and DFC are equilateral arches (see Ex. 6, p. 121). Construct a circle tangent to the four arcs AB, BC, ED, DF. (This geometric figure is the basis of various ornamental designs in Gothic windows in architecture.)



[Sug. Determine the center of the required circle by the method of intersection of loci (see p. 168). Thus find the locus of the centers of all circles tangent to the arcs AB and BC. Also find the locus of the centers of all circles tangent to the arcs AB and DF, etc.]

Ex. 19. In the diagram of Ex. 18, the figure as a whole is symmetrical with respect to what axis? Is the figure symmetrical with respect to a center? Point out four subordinate parts of the figure which have axis symmetry. Also one part which has symmetry with respect to a center.

Ex. 20. HG is a straight line, and $HA = AD = DC = CG$. ABC is an equilateral arch. H is the center of the arc ED, and G of DF. Construct a circle which shall be tangent to the arcs AB, BC, ED, and FD. (This diagram, with the dotted lines omitted, is the basis of ornamental designs in Gothic windows in architecture.)



[Sug. Denote AC by s and the radius of the required circle by r. Then in the right $\triangle HOD$, $\overline{OD}^2 = (s+r)^2 - s^2$. Also in $\triangle ODC$, $\overline{OD}^2 = (s-r)^2 - (\frac{1}{2}s)^2$. Hence find $r = \frac{3s}{16}$, and $HO = \frac{19s}{16}$.]

FORMULAS AND TABLES

SYMBOLS

a, b, c = sides of triangle ABC.
$s = \frac{1}{2}(a+b+c)$.
h_c = altitude on side c.
m_c = median on side c.
t_c = bisector of angle opposite side c.
p = perimeter.
r = radius of a circle.
d = diameter of a circle, or diagonal of a square.
C = circumference of a circle.
$\pi = \frac{22}{7}$ approx. (or 3.1416 —).
A = area.
b = base of a triangle.
h = altitude of a triangle.
b and b' = bases of a trapezoid.
a = apothem of a regular polygon.

LENGTHS OF LINES

1. In a **right triangle**, C being the right angle,
$$c^2 = a^2 + b^2. \qquad \text{§ 354.}$$

2. In an **equilateral triangle**, $h = \frac{b}{2}\sqrt{3}$.

3. In a **square**, $d = b\sqrt{2}$.

4. $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

5. $m_c = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$.

6. $t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$.

7. In **similar polygons**, $p : p' = a : a'$. § 315.

8. In **circles**, $C : C' = r : r' = d : d'$. §§ 382, 383.

9. $C = 2\pi r$, or $C = \pi d$. § 385.

10. $\text{An arc} = \frac{\text{central angle}}{180^\circ} \times \pi r$. § 387.

AREAS OF PLANE FIGURES

1. In a **rectangle**, $A = bh$. § 336.

2. In a **square**, $A = b^2$. § 336.

3. In a **parallelogram**, $A = bh$. § 338.

4. In a **triangle**, $A=\frac{1}{2}bh$. § 342.

5. In a **triangle**, $A=\sqrt{s(s-a)(s-b)(s-c)}$. Ex. 10, p. 352.

6. In an **equilateral triangle**,

$$A=\frac{b^2\sqrt{3}}{4}.$$ Ex. 4, p. 244.

7. In a **trapezoid**, $A=\frac{1}{2}(b+b')h$. § 349.

8. In a **regular polygon**, $A=\frac{1}{2}ap$. § 388.

9. In a **circle**, $A=\pi r^2$ or $A=\frac{1}{4}\pi d^2$. § 393.

10. In a **sector** of a circle, $A=\frac{1}{2}r\times$arc, § 396.

$$\text{or } A=\frac{\text{central } \angle}{360^\circ}\times\pi r^2.$$ § 396.

11. In a **segment** of a circle, $A=$sector $\pm\triangle$ formed by the chord and radii of the segment.

12. In any two **similar plane figures**.

$$A : A'=a^2 : a'^2;$$ § 352.

$$\text{also } a : a'=\sqrt{A} : \sqrt{A'}.$$ § 352, Ax. 6.

13. In two circles, $A : A'=r^2 : r'^2=d^2 : d'^2=C^2 : C'^2$. § 395.

$$r : r'=d : d'=C : C'=\sqrt{A} : \sqrt{A'}.$$ § 383, Ax. 6.

SUMMARY OF THE METRIC SYSTEM

TABLE FOR LENGTH

10 millimeter (mm.)	=1 centimeter (cm.)
10 cm.	=1 decimeter (dm.)
10 dm.	=1 meter (m.)
10 m.	=1 Dekameter (Dm.)
10 Dm.	=1 Hektometer (Hm.)
10 Hm.	=1 Kilometer (Km.)
10 Km.	=1 Myriameter (Mm.)

Similar tables are used for the unit of weight, the *gram;* for the unit of capacity, the *liter;* for the unit of land measure, the *are;* and for the unit of wood measure, the *stere.*

TABLE FOR SQUARE MEASURE

100 sq. mm. =1 sq. cm.

100 sq. cm. =1 sq. dm., etc.

TABLE FOR CUBIC MEASURE

1000 cu. mm. = 1 cu. cm.
1000 cu. cm. = 1 cu. dm., etc.

A *liter* = 1 cu. dm.
A *gram* = weight of 1 cu. cm. of water at 39.2° Fahrenheit.
An *are* = 100 sq. m.
A *stere* = 1 cu. m.

EQUIVALENTS

1 meter	= 39.37 inches
1 liter	= 1.057 liquid quarts or .9581 dry quart
1 kilogram	= 2.2046 pounds avoirdupois
1 hektare	= 2.471 acres
1 square meter	= 1550 — square inches

TABLE OF POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	**51**	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	**52**	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	**53**	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	**54**	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	**55**	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	**56**	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	**57**	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	**58**	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	**59**	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	**61**	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	**62**	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	**63**	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	**64**	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	**65**	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	**66**	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	**67**	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	**68**	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	**69**	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	**71**	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	**72**	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	**73**	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	**74**	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	**76**	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	**77**	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	**78**	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	**79**	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	**81**	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	**82**	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	**83**	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	**84**	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	**86**	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	**87**	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	**88**	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	**89**	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	**90**	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	**91**	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	**92**	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	**93**	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	**94**	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	**95**	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	**96**	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	**97**	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	**98**	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	**99**	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.642

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